Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.
2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.
3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/\sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1 & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0) = \begin{cases} F(2s,0) = f_0(2s) & 0 \le s \le 1/2 \\ G(2s-1,0 = g_0(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

Similarly $H(s,1) = (f_1 \cdot g_1)(s)$, hence $f_0 \cdot g_0 \sim f_1 \cdot g_1$.
With this, we have a well-defined $[f] \cdot [g] := [f \cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$ since $c_p \cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s) = f(1-s)$). Then $[f] \cdot [\overline{f}] = [c_p]$ and $[\overline{f}] \cdot [f] = [c_q]$.

$$H(s,t) := \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 2 \end{cases}.$$

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p) = \{\text{loops at } p\}/\sim$.
Define $[f] \cdot [g] := [f \cdot g]$.
It has an identity element $[c_p] = e$.
For any $f \in \pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f] \cdot [\overline{f}] = [\overline{f}] \cdot [f] = [c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p, q \in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma : \pi_1(X,p) \to \pi_1(X,q)$ by $[f] \mapsto [\gamma \cdot f \cdot \overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \overline{\gamma}] \cdot [\gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f \cdot \overline{\gamma} \cdot \gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\overline{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \overline{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \overline{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}} : \pi_1(X,q) \to \pi_1(X,p)$.

$$\Phi_{\overline{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\overline{\gamma}}[\gamma \cdot f \cdot \overline{\gamma}] = [\overline{\gamma} \cdot \gamma \cdot f \cdot \overline{\gamma} \cdot \gamma] = [f].$$

Induced Homomorphism

$\varphi : (X, p) \to (Y, q)$ induces

$$\begin{aligned} \varphi_* : \pi_1(X, p) &\to \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f]. \end{aligned}$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X, p) \to (Y, q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, q)$.

Proof

Let $[f] \in \pi_1(X, p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s, t) = H(f(s), t)$ such that

$$\begin{aligned} \tilde{H}(s, 0) = H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) = H(f(s), 1) = \psi \circ f(s). \end{aligned}$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples (*)

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0, 1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

IMAGE 1

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0, t]$ to $[0, 1]$ i.e.

$$\begin{aligned} \gamma_t : [0,1] &\to Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \mathrm{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\mathrm{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.

2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall\, t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1,t_2]}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}] \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ ($r \geq 0$) by

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z : |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i|\right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left|\sum_{i=1}^n |a_i z^{n-i}\right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1}z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi is})/p_t(R)}{|p_t(Re^{2\pi is})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi is})^n/R^n}{|(Re^{2\pi is})^n/R^n|} = (e^{2\pi is})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.

April 7, 2025

Definition: Retraction

Let X be a space and $A \subseteq X$ be a subset. We say that a continuous map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$. In particular, becasue $r \circ \iota_A = \mathrm{id_A}$, for $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$$

is an isomorphism. Hence $r_* : \pi(X, x_0) \to \pi(A, x_0)$ is surjective.

Corollary

There is no retraction $r : D^2 \to S^1 (= \partial D^2)$.

Proof

Suppose there is such a map r, then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \to \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

Corollary

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Proof

Suppose $\exists h : D^2 \to D^2$ without fixed points.

IMAGE 1

Define $r : D^2 \to D^2$ as the ray pictured from $h(x)$ through x to the boundary. If $x \in \partial D^2$, then by construction $r(x) = x$. Hence $r : D^2 \to S^1$ is a retraction which is a contradiction.

Corollary (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$. This carries analogously to higher dimensions.

Proof

Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. We define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. On $S^2 \subseteq \mathbb{R}^3$, we consider a loop γ at the equator by $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in [0,1]$. Because S^2 is simply connected, $g \circ \gamma : [0,1] \to S^1$ is path-homotopic to a constant loop in S^1. On the other hand, we lift $h := g \circ \gamma$ to $\tilde{h} : [0,1] \to \mathbb{R}$ with $\tilde{h}(0) = 0 \in \mathbb{R}$. Note

$$h(s+1/2) = g \circ \gamma(s+1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence $\tilde{h}(s+1/2) \in \pi^{-1}(-h(s))$ where $\pi : \mathbb{R} \to S^1$ is the covering map. Since $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$, for each $s \in [0, 1/2]$ there is an integer q_s such that $\tilde{h}(s+1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$ and

$$\tilde{h}(s+1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on s and, by continuity, q_s is a constant (call it q). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that h is homotopic to a constant loop.

Corollary (Large Fiber Lemma)

If $f : [0,1]^{n+1} \to \mathbb{R}^n$ is a continuous map, then there exist $a, b \in [0,1]^{n+1}$ such that $f(a) = f(b)$ and $|a - b| \geq 1$. Remark: if $z = f(a) = f(b)$, then the lemma says that $\operatorname{diam} f^{-1}(z) \geq 1$.

Proof

Take the sphere of radius $1/2$ in $[0,1]^{n+1}$, then by Borsuk-Ulam there exist a pair of antipodal points $a, b \in S^1$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.

Proposition

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Proof

Write $F : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g],[h])$. Then $f : [0,1] \to X \times Y$ is a loop at (x_0, y_0), $f(s) = (g(s), h(s))$, and $g : [0,1] \to X$ and $h : [0,1] \to Y$ are loops at x_0 and y_0 respectively.

Definition: Wedge Sum

Let X and Y be path-connected topological spaces. Then $X \vee Y = (X \coprod Y)/x_0 \sim y_0$
Let $\{X_\alpha\}$ be a family of such spaces. Then $\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim$.

Sketch

$$\begin{aligned} \pi_1(S^1_-, x_0) \to \pi_1(X, x_0) \quad & \text{gen} \mapsto \alpha \\ \pi_1(S^1_+, x_0) \to \pi_1(X, x_0) \quad & \text{gen} \mapsto \beta \end{aligned}$$

with $\alpha \neq \beta$, $\alpha\beta \neq \beta\alpha$. Then $\pi_1(X, x_0)$ should be $\langle \alpha, \beta \rangle$.

Definition: Free Product

Let $\{G_\alpha\}_\alpha$ be a family of groups. $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k \,:\, \text{each } g_i \text{ is a word in some } A_\alpha\}$.

Proposition

If for each α, there is a group homomorphism $\phi_\alpha : G_\alpha \to H$ then $\{\phi_\alpha\}$ induces a group homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$.

Van-Kapen Theorem

Setup

Let $X = \bigcup_\alpha A_\alpha$, each A_α open and connected where $\{A_\alpha\}$ have a common point x_0. Assume also that each $A_\alpha \cap A_\beta$ is path connected. Then $j_\alpha : A_\alpha \hookrightarrow X$ induces $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. $\{j_\alpha\}_\alpha$ induces $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ which is surjective by a similar argument as was used above for Example (*) ($S^2 = A_- \cup A_+$) applied to $X = \bigcup_\alpha A_\alpha$. Now, what is the kernel of Φ?

$$\begin{tikzcd} & A_\alpha \arrow[dr, hook] & \\ A_\alpha \cap A_\beta \arrow[ur, hook] \arrow[dr, hook] & & X \\ & A_\beta \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, hook] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, hook, "i_{\alpha\beta}"] \arrow[dr, hook, "i_{\beta\alpha}"'] & & \pi_1(X) \\
 & \pi_1(A_\beta) \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & __ \arrow[dr, hook] & \\
w \arrow[ur, hook] \arrow[dr, hook] & & __ \\
 & __ \arrow[ur, hook] &
\end{tikzcd}$$

Then $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ is NOT id in $*_\alpha \pi_1(A_\alpha)$. But through Φ, it hsould be $\mathrm{id} \in \pi_1(X, x_0)$. Hence every element in $*_\alpha \pi_1(A_\alpha)$ of the form $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ where $w \in \pi_1(A_\alpha \cap A_\beta$ is in the kernel of Φ.

Theorem (Van-Kampen)

If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, $\ker \Phi$ is the normal subgroup N generated by $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A,\ w \in \pi_1(A_\alpha \cap A_\beta)\}$. Hence $\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha, x_0))/N$.

Remarks

1. In the case that $X = A_0 \cup A_1$ with $A_0 \cap A_1$ path connected, then the intersection condition holds.

2. If $X = A_0 \cup A_1$ and $A_0 \cap A_1$ is simply connected, then $N = \{\mathrm{id}\}$ and $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$.

3. If $X = A_0 \cup A_1$ and A_1 is simply connected, then $\pi_1(X) = \pi_1(A_0)/N$ and N is the normal subgroup generated by

$$i_{01}(w)\overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

   i.e. N is the normal closure of $i_{01}(\pi_1(A_0 \cap A_1))$.

Example

IMAGE 2

For each $\alpha \in \{1, \ldots, 5\}$, let A_α be a small neighborhood of $T \cup e_1$. Every double/triple intersection is a neighborhood of T. Hence it is path continuous and we have that $\pi_1(A_\alpha) = \mathbb{Z}$. Thus $\pi_1(A_\alpha \cap A_\beta) = \mathrm{id}$, and $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N = *_1^5 \mathbb{Z}$.

Example

IMAGE 3

By Van-Kampen, $\pi_1(X) = \pi_1(A_0)$ modulo the normal closure of $i(\pi_1(A_0 \cap A_1))$. That is

$$\left\langle a, b \mid aba^{-1}b^1 \right\rangle = \mathbb{Z}^2.$$

Remark

In general, orientable M_g is the connected sum of g many toruses.

April 9, 2025

Recall: Van-Kampen Theorem

Write $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$ where N is the normal closure of $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A \cap B)$, $i_{\alpha\beta} : \pi_1(A \cap B) \to \pi_1(A)$ and $i_{\beta\alpha} : \pi_1(A \cap B) \to \pi_1(B)$.

Example

M_g is the connected sum of g many tori, and $\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$.

Example

N_g is the connected sum of g many $\mathbb{RP}^2$ (e.g. N_2 is the Klein bottle). N_g has a polygon-representation by the $2g$-gon with boundary identified through $a_1 a_1 a_2 a_2 \cdots a_g a_g$. Therefore $\pi_1(N_g) = \left\langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle$.

Abelianiztion

1. $\mathrm{Ab}(\pi_1(M_g))$ is the free abelian group generated by $\{a_1, b_1, \ldots, a_g, b_g\} = \mathbb{Z}^{2g}$.
2. $\mathrm{Ab}(\pi_1(N_g)) = \mathrm{Ab}(\left\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Corollary

None of the surfaces in $\{S^2, M_1, \ldots, M_g, \ldots, N_1, \ldots, N_g, \ldots\}$ are homotopic to each other.

Definition: Cell Complex

0-cells are points; 1-cells, e^1, are intervals; 2-cells, e^2, are disks; n-cells, e^n, are $\overline{B}^n$.
A cell complex for space X is a decomposition (assuming finite dimensions) $X = X^0 \cup X^1 \cup \cdots \cup X^n$ where X^0 is the discrete set of points (i.e. 0-cells), X^1 is the space obtained by gluing 1-cells to X^0 ($\varphi_\alpha : \partial e_\alpha^1 \to X^0$), X^2 is the space obtained by gluing 2-cells to X^1 ($\varphi_\alpha : \partial e_\alpha^2 \to X^1$), and in general X^n is obtained by gluing n-cells $\{e_\alpha^n\}_\alpha$ to X^{n-1} by $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \to X^n$.

Examples

Cell complexes need not be unique. $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$ and $S^2 = \{e^0\} \cup_\alpha \{e^2\}$.
$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$ where φ_α is given by $z \mapsto z^2$.
$\mathbb{T}^2$ is gluing e^2 to $S^1 \vee S^1$.

Theorem (Computing Fundamental Group)

Set up

Let X be a path-connected space, $Y = X \cup_\alpha e_\alpha^2$ (i.e. X is created by gluing 2-cells $\{e_\alpha^2\}_\alpha$ to X via $\phi_\alpha : \partial e_\alpha^2 \to X$). The inclusion $\iota : X \to Y$ induces $\iota_* : \pi_1(X) \to \pi_1(Y)$. Fix a base point $s_0 \in S^1$. For each α we draw a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop based at x_0. Thus $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is null-homotopic in Y (because φ_α is null-homotopic in e_α^2). That is $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \mathrm{id}$ in $\pi_1(Y)$ and is therefore in the kernel.

Theorem

Let N be the normal subgroup in $\pi_1(X)$ generated by elements of the form $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$. Then $\pi_1(Y) \cong \pi_1(X)/N$.

IMAGE 1

Example

$\mathbb{RP}^2$ is X^1 with e^2 glued to it by the map $\varphi : z \mapsto z^2$. Then $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$ where N is generated by φ. Similarly, the theorem applies to any M_g or N_g.

Definition: Deformation Retraction

For $X \subseteq Z$, $r : Z \to X$ is a retraction if $r|_X = \mathrm{id}_X$ implies $r \circ \iota = \mathrm{id}_X$. If $\iota \circ r : Z \to Z$ is homotopic to id_X, then $r_* : \pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Proof

For each α, we glue a strip S_α along γ_α. We set the base at z_0 above x_0, $Z = Y \cup_\alpha S_\alpha$. Y is a deformation retraction of Z ($\pi_1(Y) = \pi_1(Z)$).

IMAGE 2

Set $A = Z - \bigcup_\alpha \{y_\alpha\}$, where y_α is a point in e_α^2 not intersecting S_α. $B = Z - X$. A deformation retracts to X $\pi_1(A) = \pi_1(X)$. B is the union of some S_α (removing r_α) and some e_α^2 (removing ∂e_α^2). B is contractible, $\pi_1(B) = \mathrm{id}$ and $A \cap B$ is the union of strips S_α and open disks punctured at y_α. Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha \cdot \overline{\delta}_\alpha$ where δ_α runs from z_0 to x_0, call this λ_α. It suffices to show that these generate $\pi_1(A \cap B, z_0)$. Cover $A \cap B$ by $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e_\beta^2$. Then A_α is a union of strips (with trivia fundamental group) and a single punctured, open disk $e_\alpha^2 - \{y_\alpha\}$ and $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$. So $A_\alpha \cap A_\beta$ is the union of strips, equal to $A_\alpha \cap A_\beta \cap A_\gamma$ and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by $\{\lambda_\alpha\}_\alpha$. This completes the proof.

Generalization (Theorem: Part 2)

If $Y = X \cup_\alpha e_\alpha^n$ for $n \geq 3$, then $\pi_1(Y) \cong \pi_1(X)$.
This follows from the same argument where instead A_α is the union of strips and a single punctured ball $B^n - \{y_\alpha\} \simeq S^{n-1}$. So $\pi_1(A_\alpha) = \mathrm{id}$, $\pi_1(A \cap B) = \mathrm{id}$, and $\pi_1(X) \cong \pi_1(Y)$.

Theorem: Part 3

Suppose X has a cell complex $X = X^0 \cup X^1 \cup \cdots \cup X^n$. Then $\pi_1(X) \cong \pi_1(X^2)$.
The proof follows directly from part 2.

Corollary

Given any group represented by generators and relations $G = \langle g_\alpha \mid r_\beta \rangle$, there is a cell complex X_G, of dimension 2, such that $\pi_1(X_G) \cong G$.

Proof

For each g_α, we draw a circle S^1_α. Then $X^1 = \bigvee_\alpha S^1_\alpha$ has fundamental group $*_\alpha \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$. To construct X_G, for each r_β glue a 2-cell e^2_α along r_β (think of r_β as a loop in X^1). Then in $X_G := X^1 \cup_\beta e^2_\beta$ we have $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$.

April 14, 2025

Recall: Covering Spaces

Let $p : \tilde{X} \to X$, both X and $\tilde{X}$ path-connected.

1. Path-lifting: let $f : I \to X$ starting at $f(0) = x_0$. There is a unique lifting $\tilde{f}$ of f at $\tilde{x}_0 \in p^{-1}(x_0)$.

2. Homotopy-lifting: let $f_0, f_1 : I \to X$ be two paths with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose f_t is a path-homotopy between f_0 and f_1. Then there exists a unique lift $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$ at $\tilde{x} \in p^{-1}(x)$.

These come from the following: let $f_t : Y \to X$ be a homotopy between f_0 and f_1. Given $\tilde{f}_0 : Y \to \tilde{X}$ that lifts f_0, there exists a unique lifting $\tilde{f}_t$. For path-lifting, we take Y a point; for homotopy-lifting, $Y = [0,1]$.

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
I \arrow[ur, "f"] \arrow[r, "p\circ f"'] & X
\end{tikzcd}$$

Proposition 1.31 (in Hatcher)

The covering map $p : \tilde{X} \to X$ induces $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x)$.

1. p_* is injective.

2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are exactly loops at x_0 that lift to loops at $\tilde{x}_0$.

Proof of 1

Suppose $p_*[f] = \mathrm{id} \in \pi_1(X, x_0)$ where $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $[p \circ f] = \mathrm{id}$, and $[p \circ f]$ is path-homotopic to the constant loop c_{x_0}. Hence the lifting $\widetilde{p \circ f} = f$ is path-homotopic to a constant loop $c_{\tilde{x}_0}$.

Proof of 2

Let $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. $p_*[f] = [p \circ f]$, $p \circ f$ lifts to f at $\tilde{x}_0$ which is a loop at $\tilde{x}_0$.
Let f be a loop at x_0. Suppose f lifts to a loop $\tilde{f}$ at $\tilde{x}_0$ (i.e. $p \circ \tilde{f} = f$). Hence $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Example

If $p : S^1 \to S^1$ by $z \to z^2$, then $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \le \mathbb{Z} = \pi_1(S^1, 1)$.

Remark

If $p : \tilde{X} \to X$ connected, then $p^{-1}(x)$ has the same cardinality for all $x \in X$.

Proof

Fix $x_0 \in X$. Consider $\mathcal{A} = \{x \in X \;:\; p^{-1}(x) \text{ and } p^{-1}(x_0) \text{ have the same cardinality}\} \neq \varnothing$. Then $\mathcal{A}$ is open since for each $x \in \mathcal{A}$, there is a neighborhood U of x such that U is evenly covered by p (i.e. $p^{-1}(U) = \dot{\bigcup}_{\alpha\in I} V_\alpha$ where $V_\alpha \overset{p}{\cong} U$). Then $p^{-1}(x')$ has cardinality $|I|$ for all $x' \in U$. It follows, since $\mathcal{A}^c$ is open, that $\mathcal{A}$ is also closed.

Proposition

The number of sheets is given by $[\pi_1(X,x_0) : p_*(\pi_1(\tilde{X},\tilde{x}_0)]$.

Proof

Write $H = p_*(\pi_1(\tilde{X},\tilde{x}_0))$. Define $\Phi : \{H\text{-cosets in } \pi_1(X,x_0)\} \to p^{-1}(x_0)$ by $H[g] \mapsto \tilde{g}(1)$ where $\tilde{g}$ is a lift of g at $\tilde{x}_0$.
This map is well defined, since for $[h\cdot g]$ with $h \in H$, $\widetilde{h\cdot g}(1) = \tilde{g}(1)$ (because $\tilde{h}(1) = \tilde{x}_0$).
Φ is surjective. Let $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let $\tilde{g}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Define $g = p \circ \tilde{g}$, a loop at x_0. Then $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$.
Φ is injective. Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ (i.e. $\tilde{g}_1(1) = \tilde{g}_2(1)$.

IMAGE 2

Consider the loop $g_1\overline{g}_2$ in X at x_0. It lifts to $\tilde{g}_1\overline{\tilde{g}}_2$, which is a loop at $\tilde{x}_0$. This shows that $[g_1\overline{g}_2] \in H$ (i.e. $H[g_1] = H[g_2]$).

Recall (Manifolds 2)

If a smooth manifold M is non-orientable, then there is a double cover (2 sheets) $p : \hat{M} \to M$ ($\hat{M}$ connected). Consequently, $\pi_1(M)$ has a subgroup of index 2.

Definition: Locally Path-Connected

A topological space is called locally path-connected if for each $x \in X$ and every neighborhood $U \ni X$, there is a neighborhood $V \ni x$ such that $V \subseteq U$ and V is path-connected (i.e. $\forall x \in X$, there exists a local basis $\{U_\alpha\}$ at x such that each U_α is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

Proposition: Lifting Criterion

Let Y be path-connected and locally path-connected. Given a covering map $p : (\tilde{X},\tilde{x}_0) \to (X,x_0)$ and a map $f : (Y,y_0) \to (X,x_0)$, f has a lift $\tilde{f}$ at $\tilde{x}_0$ ($\tilde{f}(y_0) = \tilde{x}_0$) if and only if $f_*(\pi_1(Y,y_0)) \subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

Proof

$(\Longrightarrow)$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \pi(\tilde{X}) \arrow[d, "p_*"] \\ \pi_1(Y) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X) \end{tikzcd}$$

$$f_*\pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \leq p_*\pi_1(\tilde{X}).$$

$(\Longleftarrow)$ Let $y \in Y$, and draw a path γ from y_0 to y.

IMAGE 3

We lift $f \circ \gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y)$ as the endpoint (i.e. $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$).
This is well-defined, since $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ is a loop at x_0 and $[(f \circ \gamma) \cdot (f \circ \overline{\gamma}')] = f_*[\gamma \cdot \overline{\gamma}'] \in f_*\pi_1(Y, y_0) \leq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.
Hence $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ lifts to a loop at $\tilde{x}_0$.

IMAGE 4

Therefore $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$.
$\tilde{f}$ is continuous. Fix $f(y) \in X$ and let U be a neighborhood of $f(y)$ that is evenly covered by p. Choose a path-connected neighborhood V of y such that $f(V) \subseteq U$. We check $\tilde{f}|_V$.

IMAGE 5

Because V is path-connected, we may draw a path η in V from y to y'. Then $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$, and $\widetilde{\gamma \cdot \eta}$ is first lifting $f \circ \gamma$ at $\tilde{x}_0$ followed by lifting $f \circ \eta$ at $\tilde{\gamma}(1)$. Let $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $\widetilde{f \circ \gamma}(1) \in \tilde{U}$. Then $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$. Hence $\tilde{f}(y') = f \circ \widetilde{(\gamma \circ \eta)}(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ (i.e. $\tilde{f} = (p^{-1})|_U = f$ on V). Hence $\tilde{f}$ is continuous at y.
$\tilde{f}$ is a lift of f. In fact, $(p \circ \tilde{f})(y) = p \circ (\widetilde{f\gamma}(1)) = f(y)$.

Corollary

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If Y is simply connected, then $f_*\pi_1(Y) \leq p_*\pi_1(\tilde{X})$ always holds (i.e. we can always lift f to $\tilde{f} : Y \to \tilde{X}$ in this case).

Proposition: Unique Lifting

Given $p : \tilde{X} \to X$ and $f : Y \to X$, if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f agree at one point, then the agree everywhere on Y.

Proof

Take $\mathcal{A} = \{y \in Y \ : \ \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \varnothing$. Locally for each $y \in Y$ there exists a neighborhood V of y such that $\tilde{f} = (p^{-1})|_U \circ f$. If $y \in \mathcal{A}$, then $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take a neighborhood U of $f(y)$ that is evenly covered and $\tilde{U}$ of $\tilde{f}_1(y) = \tilde{f}_2(y)$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then on V, a path-connected neighborhood such that $f(V) \subseteq U$, $\tilde{f}_i = (p^{-1})|_U \circ f$ (i.e. $\tilde{f}_1 = \tilde{f}_2$ on V). If $y \in \mathcal{A}^c$, $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then $\tilde{U}_i \ni \tilde{f}_i(y)$ with $\tilde{U}_1 \cap \tilde{U}_2 = \varnothing$. Then on V, $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$ (ie $\tilde{f}_1$ and $\tilde{f}_2$ never agree on V). Hence $\mathcal{A} = Y$.

Remark

If $p : \tilde{X} \to X$ is a covering map, recall that a covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. This $f : \tilde{X} \to \tilde{X}$ is a lift of $p : \tilde{X} \to X$. If we fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, the lifting criterion says that $p_*\pi_1(\tilde{X}, \tilde{x}_1) \leq p_*\pi_1(\tilde{X}, \tilde{x}_2)$. In particular, if $\pi_1(\tilde{X})$ is, then this holds. Hence there is a unique lift of p (i.e. covering transformation) f such that $f(\tilde{x}_1) = \tilde{x}_2$.

April 16, 2025

Question

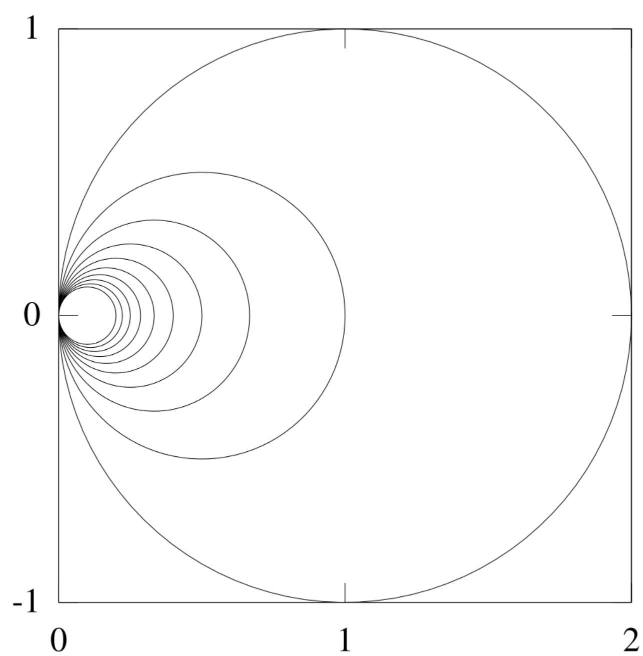
Given X path-connected and locally path-connected, when does X admit a simply connected covering space $p : \tilde{X} \to X$?

Definition: Semi-locally Simply Connected

We say that X is semi-locally simply connected if for any $x \in X$ there exists a neighborhood U such that every loop in U is null-homotopic in X. That is $\mathrm{Im}(\pi_1(U) \to \pi_1(X))$ is trivial.

Non-example

The Hawaiian earing in $\mathbb{R}^2$.



Example

The cones over the Hawaiian earing.

IMAGE 1

In fact, this is simply connected.

Example

The double Hawaiian earing with cones.

IMAGE 2

Theorem

X has a simply connected covering space (i.e. a universal covering) if and only if X is semi-locally simply connected.

Proof

($\Longrightarrow$) Let $x \in X$ and pick a neighborhood U of x that is evenly covered by p. Let f be a loop at x in U. f lifts to $\tilde{f}$ at $\tilde{x}_0$, which is a loop. Retract $\tilde{f}$ to $c_{\tilde{x}_0}$ by a path-homotopy H. Then $p \circ H$ shows that f is null-homotopic in X.
($\Longleftarrow$) We construct $\tilde{X}$ as follows: fix $x_0 \in X$ and set $\tilde{X} = \{[\gamma] \text{ path homotopies } : \gamma \text{ is a path starting at } x_0\}$. Let $\mathcal{U} = \{U : \mathrm{Im}(\pi_1(U) \to \pi_1(X)) \text{ is trivial}\}$. By assumption $\mathcal{U}$ is a basis for X. For each $u \in \mathcal{U}$ and each γ from x_0 to a point in U, we define $U_{[\gamma]} = \{\gamma \cdot \eta] : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$. Then $p : \tilde{X} \to X$ by $[\gamma] \to \gamma(1)$.
We need to check that $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$ generates a topology on $\tilde{X}$.
We need also to check that $p : U_{[\gamma]} \to U$ is bijective. It is clearly surjective, and if $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$ with η, δ paths starting at $\gamma(1)$ and staying in U. Then $\eta(1) = \delta(1)$ and, since η, δ share the same endpoints and they stay in $U_{[\gamma]}$, then $[\eta] = [\delta]$. Hence $[\gamma \cdot \eta] = [\gamma \cdot \delta]$ and p is injective.
Further, we need to check that $p : U_{[\gamma]} \to U$ is a homemorphism and that $p^{-1}(U) = \dot{\bigcup}_{[\gamma]} U_{[\gamma]}$. Hence p is a covering map.
Finally, we need to check that $\tilde{X}$ is simply connected. Recall that $p : \tilde{X} \to X$ induces an injective homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$. It suffices to show that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$. We se t$\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$. Recall also that elements in $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ are exactly the loops in X at x_0 such that they lift to loops at $\tilde{x}_0$. Suppose $[\gamma] \in p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then γ lifts to a loop $\tilde{\gamma}$ at $\tilde{x}_0 = [C_{x_0}]$. For $t \in [0,1]$, consider the path γ_t which follows γ on $[0,t]$ then stays stationary at $\gamma(t)$ for the remaining time. Then $t \mapsto [\gamma_t]$ is a path on $\tilde{X}$, $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$, and $t \mapsto [\gamma_t]$ is a lift of γ at $\tilde{x}_0 = [C_{x_0}]$. Then $t \mapsto [\gamma_t]$ is a loop (i.e. $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$) and γ is null-homotopic. This shows that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$ (i.e. $\tilde{X}$ is simply connected).

Group Actions on Fibers (Monodromy Action)

Given $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map, $\pi_1(X, x_0)$ acts on p^{-1} as follows: $p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$ by $(e, [f]) \mapsto \tilde{f}_e(1)$ where $\tilde{f}_e$ is the (unique) lift of f at $e \in p^{-1}(x_0)$. This is a right $\pi_1(X, x)$ action.
We want to check that $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$. We have that $e \cdot [f \cdot g] = \widetilde{(f \cdot g)}_e(1)$, but $\widetilde{(f \cdot g)}_e$ is the lift of f at e followed by the lift of g at the endpoint of $\tilde{f}_e$, call it $\tilde{f}_e(1) = z$. Then $\widetilde{(f \cdot g)}_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$.
This action is transitive. Given e and e', draw a path connecting them $\tilde{g}$. Under the map p, we have that $p \circ \tilde{g} = g$ which is a loop at x_0. Then $e \cdot [g] = \tilde{g}(1) = e'$.
Recall: Given a right G-set S, $G_S = \{g \in G : s \cdot g = s\}$ is the isotropy subgroup at $s \in S$.
Given $e \in p^{-1}(x_0)$, the isotropy subgroup at e is all the loops such that their lfts at e are loops (i.e. the isotropy subgroup at e is precisely $p_* \pi_1(\tilde{X}, e)$).
Recall: $G \cdot S = G/G_s$. Here, this tells us that $p^{-1}(x_0) = \pi_1(X, x_0)/p_* \pi_1(\tilde{X}, e)$. This recovers the fact that the number of sheets is equal to the index of $\mathrm{im}(p_*)$.
In particular, if $\tilde{X}$ is simply connected, then

- $\pi_1(X, x_0)$ acts freely on $p^{-1}(x_0)$ and
- the number of sheets equals the cardinality of $\pi_1(X, x_0)$.

Definition: Universal Cover

A covering space $p : \tilde{X} \to X$ is called universal if it has the universal property (i.e. for any covering space $q : Y \to X$, there is a covering map $\tilde{p} : \tilde{X} \to Y$ such that the associated diagram commutes).

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\tilde{p}"] \arrow[d, "p"'] & Y \arrow[dl, "q"] \\ X & \end{tikzcd}$$

Definition: Covering Homomorphism

Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be two covering spaces. A covering homomorphism is a map $\varphi : X_1 \to X_2$ such that the associated diagram commutes

$$\begin{tikzcd} X_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

By definition, φ is a lift of p_1.

Proposition

1. A covering homomorphism φ is uniquely determined by its value at one point.

2. For each $x \in X$, $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is $\pi_1(X, x_0)$-equivariant.

3. A covering homomorphism $\varphi : X_1 \to X_2$ is a covering map. Assuming this, the universal cover is unique.

Recall: if S_1, S_2 are right G-sets, a G-equivariant map $\varphi : S_1 \to S_2$ is a map such that the associated diagram commutes

$$\begin{tikzcd} S_1 \arrow[r, "\varphi"] \arrow[d, "\cdot g"] & S_2 \arrow[d, "\cdot g"] \\ S_1 \arrow[r, "\varphi"] & S_2 \end{tikzcd}$$

Proof of 2

Let $e \in p_1^{-1}(x)$. We need to show taht $\varphi(e) \cdot g = \varphi(e \cdot g)$. We have that $g \in \pi_1(X, x_0)$ is represented by a loop f at x_0. So $e \cdot g = e \cdot [f] = \tilde{f}_e(1) \in X_1$, and $\varphi(e \cdot g) = \varphi(\tilde{f}_e(1))$. On the left hand side, we have that $\varphi(e) \cdot g = f_{\varphi(e)}(1) \in X_2$. We need to verify that $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$ which are both lifts of f at $\varphi(e)$. But since the diagram commutes, $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$.

Uniqueness in 3

Suppose we have

$$\begin{tikzcd} X_1 \arrow[rr, shift right, "\varphi"'] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift right, "\psi"'] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

with $\varphi(e_1) = e_2$ and $\psi(e_2) = e_1$. Then $\psi \circ \varphi(e_1) = e_1$. Hence $\psi \circ \varphi = \mathrm{id}$ and, similarly, $\psi \circ \varphi = \mathrm{id}$. Hence φ is a bijection and a homemorphism.

Proof of 3

φ is surjective. Given any $e' \in X_2$, set $x_0 = p_2(e)$ and let $e \in p_1^{-1}(x_0)$ so $\varphi(e) \in p_2^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitively on $p_2^{-1}(x_0)$, there exists $g \in \pi_1(X, x_0)$ such that $e' = \varphi(e) \cdot g = \varphi(e \cdot g)$
φ is a covering map. Let V be a neighborhood of $x_0 \in X$ such that V is evenly covered by both p_1 and p_2. Let U be a component in $p_2^{-1}(V)$ that contains e_2. Then $p_1^{-1}(X) = \dot{\bigcup} U_\alpha$. U as a component in $p_2^{-1}(V)$ is both open and closed.

Hence $\varphi^{-1}(U)$ is open and closed in $p_1^{-1}(V) = \dot{\bigcup} U_\alpha$. It follows that $\varphi^{-1}(U)$ is the disjoint union of several components of $\{U_\alpha\}_\alpha$, and each component is homemorphic to V and consequently homeomorphic to U. This shows that φ is a covering map.

April 21, 2025

Recall: Lifting Criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

There exists a lift $\tilde{f}$ of f at $\tilde{x}_0$ if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

If $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (Y, y_0)$, $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ by path lifting (a right action where $e \in p^{-1}(x_0)$ by $e \cdot [\gamma] = \tilde{\gamma}_e(1)$). The isotropy subgroup at e is $p_*\pi_1(\tilde{X}, e)$.

Covering Transformations

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Write $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ for the covering group $\{\varphi : \tilde{X} \to \tilde{X} \text{ covering transformations}\}$.

1. $\varphi : \tilde{X} \to \tilde{X}$ is uniquely determined by its value at one point.

2. Given $e_1, e_2 \in p^{-1}(x)$, there is $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$. In fact, for $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ with $p_*\pi_1(\tilde{X}, e_1) \subseteq p_*\pi_1(\tilde{X}, e_2)$.

3. $\varphi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x)$ is $\pi_1(X, x)$-equivariant (i.e. $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$.

Example

Given $p : \mathbb{R} \to S^1$, what is $\mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$?
$1 \in S^1$, $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, $\forall \varphi \in \mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$, $\varphi(0) = k \in \mathbb{Z}$. Then $\varphi(x) = x + k$. In fact, the map $x \mapsto x + k$ is a covering transofmratio that agrees with φ at $0 \in \mathbb{R}$. Hence they agree everywhere (i.e. $\varphi(x) = x + k$ for all x).

Example

Given $p : S^2 \to \mathbb{RP}^2$, then $\mathrm{Aut}(S^2 \xrightarrow{p} \mathbb{RP}^2) = \{\mathrm{id}, A\}$ with A the antipodal map.

Proposition: Normal Covering

Let $\tilde{X} \xrightarrow{p} X$ be a covering map. The following are equivalent

1. There exists $x \in X$ such that $p_*\pi_1(\tilde{X}, e)$ is normal for one (thus for all) $e \in p^{-1}(x)$.

2. For every $x \in X$ and each $e \in p^{-1}(x)$, $p_*\pi_1(\tilde{X}, e)$ is normal.

3. $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ acts transitively on some (thus all) fiber $p^{-1}(x)$.

If any of these hold, we say that $p : \tilde{X} \to X$ is a normal covering.

Proof

Suppose $e, e' \in p^{-1}(x)$ with $p_*\pi_1(\tilde{X}, e)$ and $p_*\pi_1(\tilde{X}, e')$. These are the isotropy subgroups at e and e' respectively. We know also $\pi_1(X, x)$ acts transitively on $p^{-1}(x)$.
Fact: If S is a right G-set, then $G_s = \{h \in G : s \cdot h = s\}$ and $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$. So $g \cdot G_{sg} \cdot g^{-1} \in G_s$ which implies that $G_{sg} = g^{-1} \cdot G_s \cdot g$. So if G_s is normal then so is G_{sg}.

IMAGE 1

$$\begin{tikzcd}
\pi_1(\tilde{X}, e_0) \arrow[r, "\Phi_{\tilde{h}}"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, e) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[r, "\Phi_h"] & \pi_1(X, x)
\end{tikzcd}$$

commutes. Hence Φ_h maps $p_*\pi_1(\tilde{X}, e_0)$ to $p_*\pi_1(\tilde{X}, e)$, and $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$ preserves normal subgroups.

(3) implies (1)

Finally, for every $e_1, e_2 \in p^{-1}(x)$, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. This holds if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$ for every $e_1, e_2 \in p^{-1}(x)$. That is, $e_2 = e_1 \cdot \gamma$ for some $\gamma \in \pi_1(X, x)$ and $H = \gamma^{-1} H \gamma$ for every $\gamma \in \pi_1(X, x)$. So H is normal.

Remark

The (simply connected) universal cover is always normal because $\{\mathrm{id}\}$ is normal in $\pi_1(X, x)$.

Theorem

Let $p : \tilde{X} \to X$ be a covering map with $x \in X$ and $e \in p^{-1}(x)$. Then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$ where $G = \pi_1(X, x)$, $H = p_*\pi_1(\tilde{X}, e)$, and $N_G(H) = \{g \in G : g^{-1} H g = H\}$.

Special Case 1

If $p : \tilde{X} \to X$ is a normal covering, then H is normal in G. Then also $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$.

Special Case 2

If $p : \tilde{X} \to X$ is the (simply connected) universal covering, then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$.

Proof

Let S be a right G-set with transitive action and $\mathrm{Aut}_G(S)\{\varphi : S \to S$ G-equivariant bijections$\}$. Fix $s \in S$. Then $\mathrm{Aut}_G(S) \cong \frac{N_G(H)}{H}$ where $h = G_s$.
Define $\Phi : N_G(H) \to \mathrm{Aut}_G(S)$ by $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$ with $\varphi_\gamma : S \to S$ defined by

$$G_{s \cdot \gamma} = \gamma^{-1} H \gamma = H = G_s.$$

Then there exists a unique $\varphi_\gamma \in \mathrm{Aut}_G(S)$ such that $\varphi_\gamma(s) = s\cdot\gamma$.

- Lemma

  For each $s' \in S$, $s' = s\cdot\gamma'$ for some $\gamma' \in G$. Then $\varphi_\gamma(s') = \varphi_\gamma(s\cdot\gamma') = \varphi_\gamma(s)\cdot\gamma' = s\cdot\gamma\gamma'$. This is well defined. If $s' = s\cdot\gamma''$, then $s = s(\gamma\cdot\gamma''\cdot(\gamma')^{-1}\cdot\gamma^{-1})$ which implies that $\gamma\cdot\gamma''(\gamma')^{-1}\cdot\gamma^{-1} \in G_s$ and $\gamma''\cdot(\gamma')^{-1} \in G_s$.

Φ is a group homomorphism since

$$\varphi_{\gamma_1}\circ\varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s\cdot\gamma_2) = \varphi_{\gamma_1}(s)\cdot\gamma_2 = s\cdot\gamma_1\cdot\gamma_2.$$

Φ is surjective since letting $\varphi \in \mathrm{Aut}_G(S)$, it maps s to some $\varphi(s) = s' = s\cdot\gamma$ and hence $\varphi = \varphi_\gamma$.
If $\varphi_\gamma = \mathrm{id}$, then $\varphi_\gamma(s) = s$ and $\gamma \in G_s = H$. So Φ induces $\frac{N_G(H)}{H} \cong \mathrm{Aut}_G(S)$.
Take $G = \pi_1(X,x)$ and $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \to \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$ by $\varphi \mapsto \varphi|_{p^{-1}(x)}$ where H is the isotropy subgroup of the $\pi_1(X,x)$ action at e $(p_*\pi_1(\tilde{X},e))$. Then $\varphi \mapsto \varphi|_{p^{-1}(x)}$ is injective because it is uniquely determined by its value at one point.
$\varphi \mapsto \varphi|_{p^{-1}(x)}$ is surjective. Letting $\eta \in \mathrm{Aut}_g(p^{-1}(x))$ and $e_1 \in p^{-1}(x)$, we set $e_2 = \eta(e_1)$ and see that $p_*\pi_1(\tilde{X},e_1) = G_{e_1} = G_{e_2} = p_*\pi_1(\tilde{X},e_2)$. By the lifting criterion, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. Then $\varphi|_{p^{-1}(x)} = \eta$ since both are in $\mathrm{Aut}_G(p^{-1}(x))$ and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

Definition: Covering Space Action

Let X be connected and locally path connected with a group action Γ acting by homeomorphism. The quotient map $p: X \to X/\Gamma$ will be a covering map if we impose $(*)$ for all $x \in X$, there exists a neighborhood U of x such that $U \cap (g\cdot U) = \varnothing$ for each $g \in \Gamma - \{\mathrm{id}\}$. In particular, G acts freely on X. We say that a Γ-action on X is a covering space action if $(*)$ if fulfilled.

Counter-example

Consider an $\mathbb{R}$ action on $\mathbb{R}^2$ by translation. Then $U \cap (g\cdot U) \neq \varnothing$.

IMAGE 2

Remark

Assuming $(*)$, $\{g\cdot U : g \in \Gamma\}$ is a disjoint family of open sets.

Example

Take a $\mathbb{Z}$-action by $\mathbb{R}^2$ given by $\gamma(x,y) = (x+1,-y)$.

IMAGE 3

Example

S^2 with $\mathbb{Z}_2$-action $(\{\mathrm{id}, A\})$.

Theorem

If Γ acts on X as a covering space action, then $q : X \to X/\Gamma$ is a normal covering map.

Proof

Let $\overline{x} \in X/\Gamma$ and pick $x \in q^{-1}(\overline{x})$. By $(*)$, we have a neighborhood U such that $\{g \cdot U \,:\, g \in \Gamma\}$ is a disjoint collection. Let $V = q(U)$, an open neighborhood of $\overline{x}$ in X/Γ. Then $q^{-1}(V) = \{g \cdot U \,:\, g \in \Gamma\}$. Moreover, $g \cdot U \to V$ is a homeomorphism. If there exist $x', g'x' \in g \cdot U$, then $x' = h_1 \cdot u_1$ and $g' \cdot x' = h_2 \cdot u_2$. So $h_1^{-1}x' \in U$ and $h_2^{-1}g' \cdot x' \in U$ but this holds only for the identity map. So the covering map is injective.

Classifications of Covering Spaces

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{matrix}\text{isomorphism classes of covering}\\ \text{spaces } p:\hat{X}\to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups in } \pi_1(X,x)\end{matrix}\right\}$$
$$(p : \hat{X} \to X) \leftrightarrows [p_*\pi_1(\hat{X}, \hat{e})]$$

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Recall: Theorem

For X path-connected, locally path-connected and semi-locally simply path connected, Γ acts on X as a covering group action (i.e. $\forall x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for all $g \in \Gamma \setminus \{e\}$).
Then $p : X \to X/\Gamma$ is a normal covering map. Moreover $\mathrm{Aut}(X \xrightarrow{p} X/\Gamma) = \Gamma$.

Proof

$(\supseteq)$ this follows from

$$\begin{tikzcd}
X \arrow[rr, "g\cdot"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\
& X/\Gamma &
\end{tikzcd}$$

$(\subseteq)$ Let $\varphi \in \mathrm{Aut}(X \xrightarrow{p} X/\Gamma)$. That is

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\
& X/\Gamma &
\end{tikzcd}$$

commutes with φ a homeomorphism. Now let $x \in p^{-1}(\overline{x})$ where $\overline{x} \in X/\Gamma$, and let $x' = \varphi(x)$. Then $p(x) = \overline{x} = p(x')$, hence $x, x' \in p^{-1}(\overline{x})$. Hence there is $g \in \Gamma$ such that $gx = x'$. So we have

$$\varphi : X \to X\varphi(x) = x'$$
$$g : X \to Xg(x) = x'$$

so φ is equivalent to an action by g.

Theorem

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{covering maps } p{:}\hat{X}\to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups in } \pi_1(X,x_0)\end{matrix}\right\}$$

→ Assign a subgroup $H = p_*(\hat{X}, \hat{e})$ for $\hat{e} \in p^{-1}(x_0)$.
← Given a conjugacy class of subgroups, pick a subgroup H in the class.

$$H \le \pi_1(X, x_0) \cong \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence H acts naturally on $\tilde{X}$ as covering transformations. Consider $q : \tilde{X} \to \tilde{X}/H =: \hat{X}$, a normal covering map.

$$\begin{tikzcd}
\tilde{X} \arrow[dd, "p"'] \arrow[dr, "/H"] & \\
& \hat{X} \arrow[dl, dashed] \\
X &
\end{tikzcd}$$

Since $\tilde{X}/\pi_1(X, x_0) = x$, we have an induced map $\hat{p} : \hat{X} \to X$. We need to show that $\hat{p} : \hat{X} \to X$ is a covering map with $\hat{p}_*\pi_1(\hat{X}, \hat{e}) = H$ for some $\hat{e} \in \hat{p}^{-1}(x)$. Let U be a neighborhood of x such that $p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$. Then $\{\tilde{U}_\alpha\}$ is a collect iof disjoint open sets and identical to $\{g \cdot \tilde{U} \,:\, g \in \pi_1(X, x)\}$ where $\tilde{U}$ is a component of $p^{-1}(U)$. The H-action permutes the copies in $\{g \cdot \tilde{U}\} = \{\tilde{U}_\alpha\}$. Hence $q|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to \hat{X}$ is a homeomorphism. Let $\hat{U}$ be a component in $\hat{p}^{-1}(U)$. Then $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$ where $q^{-1}(\hat{U})$ is a union of components in $\dot{\bigcup}_\alpha \tilde{U}_\alpha$. Hence $\hat{U}$ is homeomorphic to U, and $\hat{p}^{-1}(U)$ is a union of components that are homemorphic to U.
Lastly, we show that $\hat{p}_*\pi_1(\hat{X}, \hat{e}_0) = H$. This is the isotropy subgroup of $\pi_1(X, x_0)$-actions at $\hat{e}_0$. $q|_{p^{-1}(x_0)} : p^{-1}(x_0) \to \hat{p}^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant (i.e. $q(e \cdot \gamma) = q(e) \cdot \gamma$, $q(e) = \hat{e}$ for $e \in \tilde{X}$). Hence γ fixes $q(e) = \hat{e}$ if and only if $q(e \cdot \gamma) = q(e)$, if and only if $e \cdot \gamma$ and e are in the same H-orbit, if and only if $\gamma \in H$.

Example 1

$X = S^1$ with $\pi_1(S^1) = \mathbb{Z}$.
$\mathbb{Z}$ has subgroups $\mathbb{Z}$, $2\mathbb{Z}$, $3\mathbb{Z}$, $\ldots$, $k\mathbb{Z}$, $\ldots$ where $k\mathbb{Z}$ corresponds to the covering map $p_k : z \mapsto z^k$.

Example 2

X the Mobius strip with $\pi_1(X) = \mathbb{Z}$ with $\pi_1(X) = \langle\gamma\rangle$ and $\gamma(x, y) = (x + 1, -y)$.
Take $H = 2\mathbb{Z} = \langle 2\gamma\rangle \le \mathbb{Z}$. Then $2\gamma(x, y) = (x + 2, y)$ and $\mathbb{R}^2/H$ is the cylinder while the cylinder modulo $\mathbb{Z}_2$ is the mobius strip.

Example 3

The Klein bottle, $K = \mathbb{R}^2/\Gamma$ with Γ genereted by $g(x, y) = (x + 1, -y)$ and $h(x, y) = (x, y + 1)$.
So $\pi_1(K) = \langle g, h\rangle$. $g^2(x, y) = (x + 2, y)$ commutes with h, so $\mathbb{Z}^2 \cong \langle g^2, h\rangle \le \pi_1(K)$ and $\mathbb{R}^2/\langle g^2, h\rangle = \mathbb{T}^2$ covers K.

Simplexes

IMAGE 1

The standard n-simplex is

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \,:\, \sum_{i=0}^{n} t_i = 1,\ t_i \ge 0, \forall i\right\}$$

$$\Delta^1 = \left\{(t_0, t_1) \in \mathbb{R}^2 \,:\, t_0 + t_1 = 1,\ t_0, t_1 \ge 0\right\}$$

IMAGE 2

$$\Delta^2 = \left\{(t_0, t_1, t_2) \in \mathbb{R}^3 \ : \ t_0 + t_1 + t_2 = 1, \ t_0, t_1, t_2 \geq 0\right\}$$

IMAGE 3

Δ^n has $(n+1)$-many faces ($(n+1)$-simplex) where the ith face is $\Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$. Let X be a topological space. A Δ-complex structure on X is a family of maps $\sigma_\alpha : \Delta^n \to X$ (n may depend on α) such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \to X$ is injective and each point is in the image of at most one of $\sigma_\alpha|_{\circ\Delta^n}$.
2. $\sigma_\alpha|_{\text{a face of } \Delta^N}$ is some $\sigma_\beta : \Delta^{n-1} \to X$ in the family.
3. $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

σ_β is $\Delta^{n-1} \overset{i\text{th face}}{\to} \Delta^n \overset{\sigma}{\to} X$.

Example

S^1 is the following iwht 1-simplex

IMAGE 4

Then the "body" of $\Delta^1 \overset{\sigma}{\to} X$ is

IMAGE 5

with $\sigma \circ \delta_0 : \Delta^0 \to X$ and $\sigma \circ \delta_1 : \Delta^0 \to X$. The boundary $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$. They define $\delta : C_n(X) \to C_{n-1}(X)$. For this example, we have $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$.
The ith face is $\delta_i : \Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.
In Hatcher's notation, the boundary is $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ where we shoild think of $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ as the ith face. So $\sigma : \Delta^n = [v_0, \ldots, v_n] \to X$. Now we have

$$\cdots \overset{\partial}{\to} C_n(X) \overset{\partial}{\to} C_{n-1}(X) \cdots$$

where $\partial^2 = 0$.

Proof

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}\right) \\
&= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}) \\
&= \sum_{j<i} (-1)^i (-1)^j \sigma_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma_{[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]} \\
&= 0
\end{aligned}$$

Homoology Associated to the Delta Complex

We have $\ker\partial \supseteq \operatorname{im}\partial$ where $\ker\partial$ are the n-cycles and $\operatorname{im}\partial$ are the n-bodies, and

$$H_n^{\delta}(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-bodies}\}} = \frac{\ker\partial}{\operatorname{im}\partial}$$

Example

For the circle, $C_1(X) = \mathbb{Z} = \langle\sigma\rangle$ and $C_0(X) = \mathbb{Z} = \langle v\rangle$. Therefore

$$\overbrace{C_2(X)}^{=0} \to \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \to 0$$

Then $H_1^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}/\{0\} = \mathbb{Z}$ and $H_0^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}$.

An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Example

For the torus, draw

IMAGE 6

So $C_0(X) = \langle v\rangle = \mathbb{Z}$, $C_1(X) = \langle a, b, c\rangle = \mathbb{Z}^3$ and $C_2(X) = \langle U, L\rangle = \mathbb{Z}^2$. Then also $\partial U = a + b - c$ and $\partial L = a + b - c$, so $\partial(U - L) = 0$ and $\ker\partial_2 = \langle U - L\rangle \cong \mathbb{Z}$. That is $H_2^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} \cong \mathbb{Z}$. Now $\partial a = 0 = \partial b = \partial c$, so $\ker\partial_1 = \langle a, b, c\rangle$ and $H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle a,b,a+b-c\rangle}{\langle a+b-c\rangle} \cong \mathbb{Z}^2$. Finally we have that $H_0^{\Delta} = \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v\rangle}{\{0\}} \cong \mathbb{Z}$.

Example

For $\mathbb{RP}^2$, draw

IMAGE 8

A$C_0(X) = \langle v, w\rangle \cong \mathbb{Z}^2$, $C_1(X) = \langle a, b, c\rangle \cong \mathbb{Z}^3$, and $C_2(X) = \langle U, L\rangle \cong \mathbb{Z}^2$. Then $\partial U = a + b + c$ while $\partial L = a + b - c$, so $\ker\partial_2 = \{0\}$ and $H_2^{\Delta}\frac{\ker\partial_2}{\operatorname{im}\partial_3} = \{0\}$. $\partial_1(a) = w - v$, $\partial_1(b) = v - w$ and $\partial_1(c) = 0$, so $\ker\partial_1 = \langle c, a - b\rangle$ and

$$H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a + b\rangle}{\langle a + b + c, a + b - c\rangle} = \langle a + b + c, c\rangle/\langle a + b + c, 2c\rangle \cong \langle c\rangle/\langle 2c\rangle \cong \mathbb{Z}^2.$$

April 28th, 2025

Recall:

For X with a Δ-complex structure, we have $H_n^{\Delta}(X)$.

Definition: Singular Simplex

A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.
The singular chain $C_n(X)$ is the free Abelian group generated by singular n-simplecies. Write

$$C_n(X) = \left\{\sum n_i\sigma_i \ : \ |\sum n_i\sigma_i| < \infty,\ n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X\right\}$$

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

While $\partial^2 = 0$ and $H_n(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$ is the singular homology.

Proposition

If $X = \coprod_\alpha X_\alpha$ with X_α connected components of X, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.

Proof

$\sigma : \Delta^n \to x$, $\operatorname{im}\sigma \subseteq X_\alpha$ for some α. So $C_n(X) = \oplus_\alpha C_n(X_\alpha)$ and $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$. Therefore $\ker\partial_n = \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)})$ and $\operatorname{im}\partial_{n+1} = \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)})$. Then $H_n(X) \cong \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)}) / \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)}) \cong \oplus_\alpha H_n(X_\alpha)$.

Proposition

Let X be a point. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof

For each n, $C_n(X)$ is generated by a single element $\sigma_n : \Delta^n \to p$ so $C_n(X) \cong \mathbb{Z}$. Then

$$\partial\sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \sum_{i=0} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow C_{n+1}(X) \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots$$

We see that

$$\partial_n = \begin{cases} \cong & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore $\ker / \operatorname{im} = 0$ or $\ker / im = \mathbb{Z}/\mathbb{Z} = 0$. Because

$$C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{0} 0$$

we have thata $H_0(X) = \ker / \operatorname{im} = \mathbb{Z}/\{0\} = \mathbb{Z}$.

Proposition

If X is path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof

Define a map $\epsilon : C_0(X) \to Z$ by $\sum n_i\sigma_i \mapsto \sum n_i$ given that $\sigma_i : \{v\} \to X$. Then ϵ is surjective. Also,

$$H_0(X) = \ker\partial_0 / \operatorname{im}\partial_1 = C_0(X)/\operatorname{im}\partial_1 = C_0(X)/\ker\epsilon \cong \operatorname{im}\epsilon \cong \mathbb{Z}$$

We claim that $\ker\epsilon = \operatorname{im}\delta$.
($\supseteq$) Let $\sigma : \Delta^1 \to X$, $\epsilon(\delta_1(\sigma)) = \epsilon(v_1 - v_0) = 1 - 1 = 0$.
($\subseteq$) Let $\sum n_i\sigma_i \in C_0(X)$ such that $0 = \epsilon(\sum n_i\sigma_i) = \sum n_i$. We fix a point $x_0 \in X$. Because X is path-connected, we can draw paths τ_i from x_0 to σ_i. Consider $\sum n_i\tau_i \in C_1(X)$, then

$$\partial(\sum n_i\tau_i) = \sum n_i(\partial\tau_i) = n_i(\sigma_i - x_0) = \sum n_i\sigma_i - \overbrace{\sum n_i}^{=0} x_0 = \sum n_i\sigma_i$$

Reduced Homology

$$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Usually written as $\tilde{H}_n(X)$, and $\tilde{H}_n(X) = H_n(X)$ if $n \geq 1$. We have that $\tilde{H}_0(X) = \ker\epsilon/\operatorname{im}\partial_1$ and $\epsilon|_{\operatorname{im}\partial_1} = 0$ so ϵ induces a map $\tilde{\epsilon} H_0(X) \hookrightarrow \mathbb{Z}$. Then $\ker\tilde{\epsilon} = \tilde{H}_0(X)$. It follows that

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a split exact sequence since $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particualr, $\tilde{H}(\text{pt}) = \{0\}$.

Remark

$$\pi_1/[\pi_1, \pi_1] \cong H_1$$

Homotopy Invariance

Suppose we have $f : X \to Y$ continuous. It induces $f_\sharp : C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. $f_\sharp$ is called a chain map and the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] & C_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Let $\sigma \in C_n(X)$ and

$$f_\sharp(\partial\sigma) = f_\sharp\left(\sum_{i=0}^{n}(-1)^i\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) = \sum_{i=0}^{n}(-1)^i(f\circ\sigma)|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \partial(f_\sharp\sigma)$$

Then $f_\sharp$ maps cycles to cycles ($\partial c = 0$, $\partial(f_\sharp c) = f_\sharp(\sigma c) = 0$) and boundaries to boundaries ($f_\sharp(\partial c) = \partial(f_\sharp c)$). So $f_\sharp$ induces $f_* : H_n(X) \to H_n(Y)$.

Theorem

If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Corollary

If $X \simeq Y$ are homotopic, then $H_n(X) \cong H_n(Y)$. $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$,

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}$$

and similarly $g_* \circ f_* = \mathrm{id}$. So f_* and g_* are isomorphisms.

Definition

Let $f, g : C.(X) \to C.(Y)$ be two chain maps. We say that f and g are chain homotopic if there is a map $p : C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g - f$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f{,}g"] \arrow[dl, "P"'] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f{,}g"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] & C_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Theorem

If $f \simeq g$ are homotopic, then

1. $f_\sharp$ and $g_\sharp$ are chain homotopic,
2. $f_* = g_*$ on homoology
3. For any n-cycle, $c \in C_n(X)$, $g(c) - f(c) = \partial P(c) + \overbrace{P(\partial c)}^{=0}$. Hence $g_*[c] = f_*[c]$.

Proof

Consider $\Delta^n \times I$, and set $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Then the following are all n-simplicies

$$\begin{aligned}
[v_0, v_1, \ldots, v_{n-1}, v_n] \\
[v_0, v_1, \ldots, v_{n-1}, w_n] \\
[v_0, v_1, \ldots, w_{n-1}, w_n] \\
\vdots \\
[v_0, w_1, \ldots, w_{n-1}, w_n] \\
[w_0, w_1, \ldots, w_{n-1}, w_n]
\end{aligned}$$

They divide $\Delta^n \times I$ into $(n+1)$-simplicies, $\{[v_0, \ldots, v_i, w_i, \ldots, w_n] : i = 0, \ldots, n\}$. Now let $F : X \times I \to Y$ be a homotopy between f and g. Consider

$$\Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{F} Y$$

and define $P : C_N(X) \to C_{n+1}(Y)$ by $\sigma \mapsto \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$. We need to check that $\partial P + P\partial = g_\sharp - f_\sharp$.

Short Exact Sequences of Chain Complexes Induce Long Exact Sequences of Homology Groups

Applications

1. Rleative homology group.
2. Meyer-Vietoris sequence.

Short Exact Sequences Induce Long Exact Sequences

Suppose we have sequences

$$
\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_* \arrow[r] \arrow[d, "i"] & \cdots \arrow[r] & & A_{n+1} \arrow[r, "\partial"] \arrow[d] & A_n \arrow[r, "\partial"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
B_* \arrow[r] \arrow[d, "j"] & \cdots \arrow[r] & & B_{n+1} \arrow[r, "\partial"] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
C_* \arrow[r] \arrow[d] & \cdots \arrow[r] & & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
0 & & & 0 & 0 & 0 &
\end{tikzcd}
$$

So H induces a long exact sequence

$$\xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C)$$

$$\xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$, our connecting homomorphism, for $c \in C_n$. Then we have that the following commutes

$$
\begin{tikzcd}
& a \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i"] & {} \arrow[d, "i"] \\
b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & 0 \\
c \arrow[r, mapsto] & 0 &
\end{tikzcd}
$$

So a is a cycle. We need to show that $\partial a = 0$. Note that $i(\partial a) = \partial(ia) = \partial(\partial b) = 0$. Because i is injective, $\partial a = 0$. ∂ is well defined since

- choice of a: i is injective
- choice of b: suppose $b' \in B_n$ such tha t$j(b') = j(b) = c$. Then $b-b'$ satisfies $j(b-b') = 0$ and $b-b' \in \ker j = \operatorname{im} i$ (i.e. there exists $a' \in A_n$ such that $i(a') = b - b'$, so $b' = b + i(a')$. Then

$$\begin{tikzcd}
a' \arrow[r, mapsto] \arrow[d, mapsto] & \partial a' \\
b - b' \arrow[d, mapsto] & \\
0 &
\end{tikzcd}$$

So $a + \partial a'$ satisfies

$$i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial(i a') = \partial b'$$

and

$$\begin{tikzcd}
 & a + \partial a' \arrow[d, mapsto] \\
b' \arrow[r, mapsto] \arrow[d, mapsto] & \partial b' \\
c &
\end{tikzcd}$$

since $[a + \partial a'] = [a]$.

- We need to check choice of c, but we will skip this.
- We need to check that δ is a homomorphism, which follows from the definitions.
- Finally, check that the induced long sequence is exact. We will check only exactness about $H_n(C)$ (i.e. $\operatorname{im} j_* = \ker \delta$).

  $\operatorname{im} j_* \subseteq \ker \delta$: $\delta(j_*[b]) = 0$ because

$$\begin{tikzcd}
 & 0 \arrow[d, mapsto] \\
b \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "j"] & 0 \\
j(b) &
\end{tikzcd}$$

$\ker \delta \subseteq \operatorname{im} j_*$: Suppose $[c] \in H_n(C)$ such that $\partial[c] = 0$, then

$$\begin{tikzcd}
a' \arrow[r, mapsto, "\partial"] & a = \partial a' \arrow[d, mapsto] \\
b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \\
c &
\end{tikzcd}$$

Consider $b - i(a')$, then $j(b - i(a')) = j(b) - \overbrace{j \circ i(a')}^{=0} = j(b) = c$. So $[c] = j_*[b - i(a')] \in \operatorname{im} j_*$. This is a cycle, since $\partial(b) - \partial(i(a')) = \partial b - i(\partial a') = \partial b - \partial b = 0$.

April 30, 2025

Recall

1. if $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$.

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp = g_\sharp"] \arrow[dl, "P"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp = g_\sharp"] \arrow[dl, "P"'] & \cdots \\ \cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & \cdots \end{tikzcd}$$

$$\partial P + P\partial = f_\sharp - g_\sharp.$$

1. Short exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Relative Homoology Group

Setup: $A \subseteq X$, A closed and non-empty. Then

$$C_n(A) = \{c \in C_n(X) \,:\, c = \sum n_i\sigma_i,\ \operatorname{im}\sigma_i \subseteq A\}.$$

Define $C_n(X,A) = C_n(X)/C_n(A)$ such that

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$

is a short exact sequence. Then $C_*(X,A)$ is a chain complex

$$\cdots \xrightarrow{\partial} C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \xrightarrow{\partial} \cdots$$

with $\partial^2 = 0$. Note that $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$. Hence it induces $\partial : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$. It gives homoology groups $H_n(X,A) = \ker\partial_n / \operatorname{im}\partial_{n+1}$ and induces a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Remarks

1. the elements in $H_n(X,A)$ are represented by relative cycles (i.e. $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$).

2. A relative cycle α is trivial in $H_n(X,A)$ means α is a "relative boundary" (i.e. $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$).

$\partial : H_n(X,A) \to H_{n-1}(A)$ is defined by $[\alpha] \mapsto [\partial\alpha]$

$$\begin{tikzcd} & \partial\alpha \arrow[d, "i"] \\ \alpha \in C_n(X) \arrow[r, mapsto, "\partial"] \arrow[d, "j"] & \partial\alpha \in C_{n-1}(A) \\ \alpha \in C_n(X,A) & \end{tikzcd}$$

We can also define the relative version.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & C_0(A) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r, "\partial"] & C_0(X) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r, "\partial"] & C_0(X,A) \arrow[r, "0"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
\sum n_i\sigma_i \arrow[r] \arrow[d] & \sum n_i \arrow[d, "="] \\
\sum n_i\sigma_i \arrow[r, mapsto] & \sum n_i
\end{tikzcd}$$

$$\tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X,A)$$

$$\xrightarrow{\partial}$$

$$\longrightarrow \tilde{H}_0(A) \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X,A)$$

$$\longrightarrow 0$$

Example

$H_n(X,X) = 0$ for all n, because $C_n(X,X) = C_n(X)/C_n(X) = \{0\}$. So $H_n(X,X_0) \cong \tilde{H}_n(X)$

$$\overbrace{\tilde{H}_n(X_0)}^{=0} \longrightarrow \tilde{H}_n(X) \xrightarrow{\cong} H_n(X,X_0)$$

$$\xrightarrow{\partial} \overbrace{\tilde{H}_{n-1}(X_0)}^{=0} \longrightarrow \cdots$$

Fact

$H_n(X,A) \cong \tilde{H}_n(X/A)$ if (X,A) is a "good" pair (i.e. there exists a neighborhood V of A which deformation retracts to A).

Example

$(X,A) = (D^n, \partial D^n)$ is a good pair, so $H_i(X,A) \cong \tilde{H}_i(D^n/\partial D^n) = \tilde{H}_i(S^n)$. This give a long exact sequence

$$\tilde{H}_i(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_i(D^n)}^{=0} \longrightarrow H_i(X,A)$$

$$\xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_{i-1}(D^n)}^{=0} \longrightarrow \cdots$$

and $\tilde{H}_{i-1}(S^{n-1}) \cong H_i(D^n, \partial D^n) \cong \tilde{H}_i(S^n)$. We conclude that $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$.
For $n = 0$, S^0 is two points, $\tilde{H}_0(S^0) = \mathbb{Z}$, and $\tilde{H}_i(S^0) = \tilde{H}_i(\text{pt}) \oplus \tilde{H}_i(\text{pt}) = 0$ for each $i \geq 1$.
For $n = 1$, $\tilde{H}_1(S^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_0(S^1) = 0$.
For $n = 2$, $\tilde{H}_2(S^2) \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$, $\tilde{H}_1(S^2) \cong \tilde{H}_0(S^1) = 0$ and $\tilde{H}_0(S^2) = 0$.
So $\tilde{H}_i(S^n$ is $\mathbb{Z}$ when $i = n$ and 0 otherwise.

Induced Maps on Pairs

Write $f : (X,A) \to (Y,B)$ for a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. Then $f_\sharp : C_n(X) \to C_n(Y)$ ($f_\sharp : C_n(A) \to C_n(B)$) induces $f_\sharp : C_n(X,A) \to C_n(Y,B)$ a chain map $\partial f_\sharp = f\sharp\partial$. This induces $f_* : H_n(X,A) \to H_n(Y,B)$.

Proposition

Given $f, g : (X,A) \to (Y,B)$ which are homotopic through maps between pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r] \arrow[d] \arrow[dl, "p"'] & C_n(X,A) \arrow[r] \arrow[d] \arrow[dl, "p"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y,B) \arrow[r] & C_n(Y,B) \arrow[r] & \cdots
\end{tikzcd}$$

such that $\partial P + P\partial = g_\sharp - f_\sharp$ (i.e. $f_* = g_*$). $P : C_n(X) \to C_{n+1}(Y)$ maps $C_n(A)$ to $C_{n+1}(B)$. P defined by $P(\sigma)\sum(-1)^i F \circ (0 \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_j]}$

$$\Delta^n \times I \xrightarrow{0\times\mathrm{id}} X \xrightarrow{F} Y$$

If $\sigma : \Delta^n \to A$, then $P(\sigma) : \Delta^{n+1} \to B$.

Excision

Given a good pair (X,A), $H_n(X,A) \cong \tilde{H}_n(X/A)$.
Suppose we have $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^\circ$ (i.e. the closure of Z is in the interior of A). Then $H_n(X,A) \cong H_n(X-Z, A-Z)$. Equivalently, if $B = X - Z$ then $A \cap B = A - Z$ and $\overline{Z} \subseteq A^\circ \implies A^\circ \cup B^\circ = X$. If A and B satisfy $A^\circ \cup B^\circ = X$, then by excision $H_n(X,A) \cong H_n(B, A\cap B)$.

Remark

If X has a Δ-complex structure such that A, $X-Z$ and $A-Z$ are subcomplexes, then we claim that $C_n^\Delta(X,A) = C_n^\Delta(X-Z, A-Z)$ (and $H_n^\Delta(X,A) = H_n^\Delta(X-Z,A-Z)$. In fact, consider $\varphi : C_n^\Delta(X-Z) \to C_n^\Delta(X)/C_n^\Delta(A)$ which factors through

$$C_n^\Delta(X-Z) \xhookrightarrow{\iota} C_n^\Delta(X) \longrightarrow C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A)$$

Then φ is surjective, $\ker\varphi = C_n^\Delta(A-Z)$ and

$$C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A) = C_n^\Delta(X-Z)/\ker\varphi = C_n^\Delta(X-Z,A-Z)$$

Proof

Let $\{U_\alpha\}_\alpha = \mathcal{U}$ be a collection of subsets such that $\{U_\alpha^\circ\}_\alpha$ is an open cover of X (it will suffices to consider $\mathcal{U} = \{A,B\}$). Write

$$C_n^{\mathcal{U}}(X) = \left\{\sum n_i\sigma_i \in C_n(X) \,:\, \operatorname{im}\sigma_i \subseteq U_j^\circ \text{ for some } j\right\}.$$

Then $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$. The chain complex $C_*^{\mathcal{U}}(X)$ gives homoology groups $H_*^{\mathcal{U}}(X)$.

Proposition

$\iota : C_n^{\mathcal{U}} \to C_n(X)$ induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.
The sketch of this proof is to construct a map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ by subdivision. That is, if the simplex $\sigma : \Delta^n \to X$ does not sit inside any U_α we may subdivide into further simplices that do. Then $\rho \circ \iota = \mathrm{id}$ and $\iota\circ\rho$ is chain homotopic to the identity.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r] \arrow[d, "\iota\circ\rho"] \arrow[dl, "D"'] & C_{n-1}(X) \arrow[r] \arrow[d] \arrow[dl, "D"'] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r] & C_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where $D : C_{n-1}(X) \to C_n(X)$ such that $\partial D + D\partial = \mathrm{id} - \iota\circ\rho$ which implies $(\iota\circ\rho)_* : H_n(X) \to H_n(X)$ is the identity map. There also exists a relative version. For simplicty, say $\mathcal{U} = \{A,B\}$ and denote $C_n^{\mathcal{U}}(X) \stackrel{\Delta}{=} C_n(A+B)$ so we have $H_n(A+B,A) \cong H_n(X,A)$.

Proof Continued

We have that $H_n(A+B,A) \cong H_n(X,A)$ (proof in Hatcher).
The left hand side comes from the chain complex of

$$C_n(A+B,A) = C_n(A+B)/C_n(A) = C_n(B)/C_n(A\cap B) = C_n(B,A\cap B)$$

so $H_n(A+B,A) = H_n(B,A\cap B)$.

Proposition

Let (X,A) be a good pair. Then the quotient map $q:(X,A)\to(X/A,A/A)$ induces an isomorphism $q_*: H_n(X,A) \xrightarrow{\sim} H_n(X/A,\text{pt}) \cong \tilde{H}_n(X/A)$.

Proof

Let V be a neighborhood of A which deformation retracts to A.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "(1)"] \arrow[d, "q_*"] & H_n(X,V) \arrow[r, "\sim"] & H_n(X-A,V-A) \arrow[d, "\sim"] \\
H_n(X/A,A/A) \arrow[r, "(2)"] & H_n(X/A,V/A) & \substack{H_n((X-A)/A,(V-A)/A) \\ = \\ H_n(X/A-A/A,V/A-A/A)} \arrow[l, "\text{excision}"']
\end{tikzcd}$$

It remains to show that (1) and (2) are isomorphisms. For (2), V/A deformation retracts to A/A in X/A. So consider the triple $A\subseteq V\subseteq X$. It induces a short exact sequence

$$0 \longrightarrow \substack{C_n(V,A) \\ = \\ C_n(V)/C_n(A)} \xrightarrow{i} \substack{C_n(X,A) \\ = \\ C_n(X)/C_n(A)} \xrightarrow{j} \substack{C_n(X,V) \\ = \\ C_n(X)/X_n(V)} \longrightarrow 0$$

So $\ker j = \operatorname{im} i$, and this induces a long exact sequence

$$\longrightarrow \overbrace{H_n(V,A)}^{=0} \longrightarrow H_n(X,A) \xrightarrow{\sim} H_n(X,V)$$

$$\xrightarrow{\partial} \overbrace{H_{n-1}(V,A)}^{=0} \longrightarrow$$

where the terms zero since V deformation retracts to A.

May 5, 2025

Recall

For $A\subseteq X$, we have

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) = C_*(X)/C_*(A) \longrightarrow 0$$

which induces

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Also, we have excision where

1. if $Z\subseteq A\subseteq X$ such that $\overline{Z}\subseteq A^\circ$, then $H_n(X-Z,A-Z) = H_n(X,A)$.

2. if (X,A) is a good pair, i.e. A has a neighborhood V such that V deformation retracts to A, then $H_n(X,A) = \tilde{H}_n(X/A)$.

Simplicial and Singular Homology

Goal: given X with Δ-complex structure, $H_n^{\Delta}(X) \cong H_n(X)$.

Example

$H_n(D^n,\partial D^n) \cong \tilde{H}_n(D^n/\partial D^n) = \tilde{H}_n(S^n) \cong \mathbb{Z}$. We can construct a generator for this $\mathbb{Z}$. We consider $H_n(\Delta^n,\partial\Delta^n)$ and claim that it is generated by $i_n : \Delta^n \to \Delta^n$ as the identity map. We prove by induction, first observing that $n = 0$ is good. Then suppose $n-1$ and let $\Lambda \subseteq \Delta^n$ be the space obtained by removing a face from the boundary $\partial\Delta^n$. Then take

$$H_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial} H_n(\partial\Delta^n,\Lambda) \xleftarrow[(2)]{} H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$$

Consider the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ and the short exact sequence on the chain level

$$0 \longrightarrow C_\bullet(\partial\Delta^n,\Lambda) \xrightarrow{i} C_\bullet(\Delta^n,\Lambda) \xrightarrow{j} C_\bullet(\Delta^n,\partial\Delta^n) \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \longrightarrow H_n(\partial\Delta^n,\Lambda) \longrightarrow \overbrace{H_n(\Delta^n,\Lambda)}^{=0} \longrightarrow H_n(\Delta^n,\partial\Delta^n) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(\partial\Delta^n,\Lambda) \longrightarrow \overbrace{H_{n-1}(\Delta^n,\lambda)}^{=0} \longrightarrow \cdots$$

since Δ^n deformation retracts to Λ, $H_*(\Delta^n,\Lambda) = 0$. Hence $H_n(\Delta^n,\partial\Delta^n) \underset{\partial}{\cong} H_{n-1}(\partial\Delta^n,\Lambda)$.

For (2), let Δ^{n-1} be the face that is not in Λ. Then $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ induces a homeomorphism $\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$. Hence $(\partial\Delta^n,\Lambda)$ Is a good pair, and

$$H_{n-1}(\partial\Delta^n,\Lambda) \cong \tilde{H}_{n-1}(\partial\Delta^n/\Lambda) \cong \tilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$$

We have

$$\begin{tikzcd}
 & \partial i_n \in C_{n-1}(\partial\Delta^n,\Lambda) \arrow[d] \\
i_n \in C_n(\Delta^n,\Lambda) \arrow[r, "\partial"] \arrow[d] & \partial i_n \in C_{n-1}(\Delta^n,\partial\Delta^n) \\
i_n \in C_n(\Delta^n,\partial\Delta^n) &
\end{tikzcd}$$

so $\delta^{-1} : [\partial i_n] \mapsto [i_n]$. Through the isomorphism $H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \cong H_n(\Delta^n,\partial\Delta^n)$, $[i_n]$ is identified with $[\partial i_n]$ for $i_n : \Delta^n \to \Delta^n$. Hence $[\partial i_n]$ is $[\pm i_{n-1}]$ in $H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$.

Corollary

Let $\bigvee_\alpha X_\alpha$ by identifying $x_\alpha \in X_\alpha$ for each α. Suppose (X_α, x_α) is a good pair for each α. Then $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(\bigvee_\alpha X_\alpha)$.

Proof

Consider the good pair $(X,A) := \left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\right)$ where $X/A = \bigvee_\alpha X_\alpha$ such that

$$\tilde{H}_n\left(\bigvee_\alpha X_\alpha\right) \cong H_n(X,A) \cong \bigoplus_\alpha H_n(X_\alpha, x_\alpha) = \bigoplus_\alpha \tilde{H}_n(X_\alpha).$$

Theorem

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets. If U and V are homeomorphic, then $m = n$.

Proof

Let $x \in U$. By excision,

$$H_i(U, U - \{x\}) \cong H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}$$

where we note that $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ is not a good pair. However, it still induces a long exact sequence

$$\longrightarrow \tilde{H}_i(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_i(\mathbb{R}^m)}^{=0} \longrightarrow H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

$$\longrightarrow \tilde{H}_{i+1}(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_{i-1}(\mathbb{R}^m)}^{=0} \longrightarrow \cdots$$

Hence

$$H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}.$$

If U and V are homemorphisms, then $H_i(U, U - \{x\}) \cong H_i(V, V - \{\varphi(x)\})$ and $m = n$.

Naturality of Long Exact Sequences of Pairs

$f : (X,A) \to (Y,B)$ with $f(A) \subseteq B$,

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d, "f_\sharp"] & C_\bullet(X) \arrow[r] \arrow[d, "f_\sharp"] & C_\bullet(X,A) \arrow[r] \arrow[d, "f_\sharp"] & 0 \\
0 \arrow[r] & C_\bullet(B) \arrow[r] & C_\bullet(Y) \arrow[r] & C_\bullet(Y,B) \arrow[r] & 0
\end{tikzcd}$$

commutes. Then the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\delta"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{array}{ccc} & & \partial\alpha \in C_{n-1}(A) \\ & & \downarrow \\ \alpha \in C_n(X) & \xrightarrow{\partial} & \partial\alpha \in C_{n-1}(X)) \\ \downarrow & & \\ \alpha \in C_n(X,A) & & \end{array}$$

So $\delta : [\alpha] \to [\partial\alpha]$ and

$$f_*(\delta[\alpha]) = f_*[\partial\alpha] = [f_*(\partial\alpha)] = [\partial f_*(\alpha)] = \delta(f_*[\alpha]).$$

Recall: the Five Lemma

$$\begin{tikzcd} \cdots \arrow[r] & A \arrow[r] \arrow[d, "\cong"] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots \end{tikzcd}$$

implies that $C \cong C'$.

Equivalence Between Simplicial and Singular Homology

Given X with a finite dimensional Δ-complex structure, then $C_n^\Delta(X) \hookrightarrow C_n(X)$ induces an isomorphism $H_n^\Delta(X) \cong H_n(X)$.

Proof

Suppose it holds for all (X, Δ) with dimension less than $k-1$. We condier the k-dimensional case.
Let X^i be the i-skeleton of X. Note that $X^k = X$, so the pair (X^k, X^{k-1}) induces a long exact sequence

$$\begin{tikzcd} H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "(1)"] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "(2)"] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "(3)"] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "(4)"] & H_{n-1}^\Delta(X^{k-1}) \arrow[d, "(5)"] \\ H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \end{tikzcd}$$

We have that (2) and (5) are isomorphisms per our inductive assumption. Note also that $C_n^\Delta(X^k) = 0$ for $n \geq k$, so

$$C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k)/C_n^\Delta(X^{k-1}) = \begin{cases} C_n^\Delta(X^k) & k = n \\ 0 & n < k \end{cases}.$$

So the chain complex $C_\bullet^\Delta(X^k, X^{k-1})$ is

$$0 \longrightarrow 0 \longrightarrow C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k) \longrightarrow 0 \longrightarrow 0$$

and $H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} C_k^\Delta(X^k) & k = n \\ 0 & k \neq n \end{cases}$. Now consider $\Phi : \left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k\right) \to (X^k, X^{k-1})$. It induces a homemorphism $X^k/X^{k-1} \cong \left(\coprod_\alpha \Delta_\alpha^k\right)/\left(\coprod_\alpha \partial\Delta_\alpha^k\right)$. So \(

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n(X^k, X^{k-1}) \cong \tilde{H}_n\left(\left(\coprod_\alpha \Delta_\alpha^n\right)/\left(\coprod_\alpha \partial\Delta_\alpha^k\right)\right) \cong H_n\left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k\right) \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$$

where each $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ is generated by $i_\alpha^k : \Delta_\alpha^k \to \Delta_\alpha^k$ (the identity map) if $n = k$ or $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ when $n \neq k$. Finally, we observe that

$$C_k^\Delta(X^k) \cong \bigoplus_\alpha \left\langle i_\alpha^k \right\rangle \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k).$$

So (1) and (4) are isomorphisms and, by the five lemma, (3) is an isomorphy as well.

Remark

$H_n^\Delta(X,A) \cong H_n(X,A)$ if X has a Δ-complex structure and $A \subseteq X$ is a sub-complex.

$$\begin{tikzcd} H_n^\Delta(A) \arrow[r] \arrow[d, "(1)"] & H_n^\Delta(X) \arrow[r] \arrow[d, "(2)"] & H_n^\Delta(X,A) \arrow[r] \arrow[d, "(3)"] & H_{n-1}^\Delta(A) \arrow[r] \arrow[d, "(4)"] & H_{n-1}^\Delta(X) \arrow[d, "(5)"] \\ H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \end{tikzcd}$$

where (1), (2), (4), (5) are isomorphisms, so we have the conclusion by the five lemma.

May 7, 2025

Definition: Degree

Let $f : S^n \to S^n$ which induces $f_* : H_n(S^n) \to H_n(S^n)$ (i.e. $\mathbb{Z} \to \mathbb{Z}$). Hence f_* is multiplication by some integer $d \in \mathbb{Z}$. Define $\deg(f) = d$.

Properties

1. $\deg(\mathrm{id}) = 1$.

2. If $f, g : S^n \to S^n$ are homotopic, then $f_* = g_*$ thus $\deg(f) = \deg(g)$.

3. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, because $(f \circ g)_* = f_* \circ g_*$. In particular, if $f \circ g \simeq \mathrm{id}_{S^n}$ then $\deg(f) \cdot \deg(g) = \deg(f \circ g) = 1$ and $\deg(f) = \pm 1$.

4. Suppose $f : S^n \to S^n$ is not surjective, say $x_0 \in S^n \backslash \operatorname{im} f$. Then $f : S^n \to S^n \backslash \{x_0\} \cong \mathbb{R}^n$. So f is $S^n \xrightarrow{f} S^n \backslash \{x_0\} \xrightarrow{\iota} S^n$ and

$$H_n(S^n) \longrightarrow \overbrace{H_n(S^n \setminus \{x_0\})}^{=0} \longrightarrow H_n(S^n)$$

So $f_* : H_n(S^n) \to H_n(S^n)$ is the zero map (i.e. $\deg(f) = 0$).

1. $f : S^n \to S^n$ a reflection has degree -1. In general, if we take two copies of Δ^n glued along corresponding edges by the identity map then we get S^n. Then $H_n^\Delta(S^n)$ has a generator $U - L$, and reflection of f maps $U - L$ to $L - U$ (i.e. $f_* : \mathbb{Z} \to \mathbb{Z}$ is $1 \mapsto -1$).

2. $f : S^n \to S^n$ an antipodal map $(-\mathrm{id})$ which sends $(x^1, \dots, x^{n+1}) \mapsto (-x^1, \dots, -x^{n+1})$ has $\deg(-1\,\mathrm{id}) = (-1)^{n+1}$.

3. Theorem (Hopf) if $f, g : S^n \to S^n$ have the same degree, then $f \simeq g$.

4. If $f : S^n \to S^n$ has no fixed points, then $f \simeq -\mathrm{id}$ and $\deg(f) = (-1)^{n+1}$. Proof: if $x \neq f(x)$, then the segment $(1-t)f(x) + t(-x)$ does not pass through $0 \in \mathbb{R}^{n+1}$. Consider $f_t(x) = \frac{(1-t)f(x)+t(-x)}{||(1-t)f(x)+t(-x)||}$ where $f_0(x) = f(x)$ and $f_1(x) = -x$ show that $f_t(x)$ gives a homotopy between f and $-\mathrm{id}$.

5. S^n has a continuous, non-vanishing vector field if and only if n is odd. Proof: $(\Longleftarrow)$ say $n = 2k-1$ such that $S^n \subseteq \mathbb{R}^{2k}$. Define $V(x_1, \dots, x_{2k}) = (-x_2, x_1, -x_4, x_3, \dots)$. Then $V(\vec{x}) \perp \vec{x}$. $(\Longrightarrow)$ Think of $V(\vec{x})$ starting at $\vec{x}$

and without loss of generality that $||V(\vec{x})|| = 1$. Consider $f_t(x) = (\cos t)\vec{x} + (\sin t)V(\vec{x})$ where $f_\pi(x) = -x$ and $f_0(x) = x$ such that $\{f_t\}$ is a homotopy between id and $-\mathrm{id}$. Hence $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}$ and n is odd.

6. If n is even, then $\mathbb{Z}_2$ is the only non-trivial group that can act freely on S^n. For example, S^1 acts on S^3 freely if we consider $(z_1, z_2) \in S^3 \subseteq C^2$ and $\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$. Proof: suppose $G \neq \mathrm{id}$ acts freely on S^n. Consider $\deg : G \to \mathbb{Z}$ where $\mathrm{im}(\deg) \subseteq \{\pm 1\} \subseteq \mathbb{Z}$ and for $g \neq e$ then $\deg(g) = (-1)^{n+1} = -1$. Then $G/\ker \cong \mathrm{im} = \{-1, 1\}$ since $\ker = \{e\}$. Hence $G \cong \mathrm{im} = (\{\pm 1, \cdot\}) = \mathbb{Z}_2$.

Theorem

Below, we assume that S^n has a point y such that $f^{-1}(y) = \{x_1, \dots, x_m\}$ is a finite set. If f is smooth, then by Sard's theorem we may pick a regular point y. Then $f^{-1}(y)$ is an embedded submanifold of dimension zero (i.e. $f^{-1}(y)$ is a collection of finitely many points). That is, when f is smooth this assumption holds automatically.
For each $i = 1, \dots, m$, we choose a small ball U_i about x_i and a ball V about y such that $f(U_i) \subseteq V$. The pair $(S^n, S^n \setminus \{x\})$ induces

$$\cdots \longrightarrow \overbrace{H_n(S^n \setminus \{x\})}^{=0} \longrightarrow H_n(S^n) \xrightarrow{j} H_n(S^n, S^n \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(S^n \setminus \{x\})}^{=0} \longrightarrow \cdots$$

The pair $(U, U \setminus \{x\})$ gives

$$\cdots \longrightarrow H_n(U \setminus \{x\}) \longrightarrow \overbrace{H_n(U)}^{=0} \longrightarrow H_n(U, U \setminus \{x\}) \xrightarrow{\delta} H_{n-1}(U \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(U)}^{=0} \longrightarrow \cdots$$

and we observe that $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ by excision.

$$\begin{tikzcd}
 & \overbrace{H_n(U_i, U_i \setminus \{x_i\})}^{=\mathbb{Z}} \arrow[r, "f_*"] \arrow[dl, "excision"'] \arrow[d, "k_i"] & \overbrace{H_n(V, V \setminus \{y\})}^{=\mathbb{Z}} \arrow[d, "excision"] \\
H_n(S^n, S^n \setminus \{x_i\}) & \overbrace{H_n(S^n, S^n - f^{-1}(y))}^{=\mathbb{Z}^m} \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[ul, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "j"]
\end{tikzcd}$$

We have that $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$ is $\mathbb{Z} \to \mathbb{Z}$ and hence it gives an integer. We call this the local degree $\deg(f|_{x_i})$.
Theorem: $\deg(f) = \sum_{i=1}^m \deg(f|_{x_i})$.
Write

$$H_n(S^n, S^n - f^{-1}(y)) \underset{\text{excision}}{\cong} H_n\left(\coprod_i U_i, \coprod_i (U_i \setminus \{x_i\}\right) \cong \bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}^m.$$

then $k_i : H_n(U_i, U_i \setminus \{x_i\}) \to \bigoplus_i H_n(U_i, U_i \setminus \{x_i\})$ by $1 \mapsto (0, \dots, 0, 1, 0, \dots, 0) =: e_i$. Consider the triple $S^n - f^{-1}(y) \subseteq S^n \setminus \{x_i\} \subseteq S^n$ which induces

$$0 \longrightarrow C_\bullet(S^n \setminus \{x_i\}, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus \{x_i\}) \longrightarrow 0$$

So we have $p_i : H_n(S^n, S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{x_i\})$. Then

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[d, "k_i"] \\ \mathbb{Z} & \mathbb{Z}^m \arrow[l, "p_i"] \end{tikzcd}$$

commutes and $1 = p_i(k_i(1)) = p_i(e_i)$, hence p_i is the projection to the i-th component. Similarly

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z}^m \arrow[l, "p_i"'] \\ & \mathbb{Z} \arrow[ul, "\mathrm{id}"] \arrow[u, "j"'] \end{tikzcd}$$

commutes so $1 = p_i(j(1))$ and the i-th component of $j(1)$ is 1 (i.e. $j(1) = (1, 1, \ldots, 1) \in \mathbb{Z}^m$. Then $\deg(f|_{x_i}) = f_*(k_i(1)) = f_*(e_i)$. Finally,

$$\deg f = f_*(1) = f_*(j(1)) = f_*\left(\sum e_i\right) = \sum f_*(e_i) = \sum \deg(f_*|_{x_i})$$

Remark

If f is smooth and y is a regular value, then we can pick U_i and V such that each $f|_{U_i} : U_i \to V$ is a diffeomorphism. Hence $\deg(f|_{x_i}) = \pm 1$.

Example

If $f : S^1 \to S^1$ by $z \mapsto z^k$, $f^{-1}(1)$ has k many points (viz. the roots of unity). $f|_{U_i} : U_i \to V$ is diffeomorphic (by rotation and scaling) and $\deg(f|_{x_i}) = 1$. $\deg(f) = \sum \deg(f|_{x_i}) = k$.

IMAGE 1

Definition: Suspension of a Space

Recall that the cone of X is $C(X) = X \times I / X \times \{1\}$.

IMAGE 2

The suspension of X is $S(X) = C(X)/X \times \{0\}$.

IMAGE 3

Examples

$S(S^1) = S^2$. In general $S(S^n) = S^{n+1}$.

Definition: Suspension of a Map

$f : X \to Y$ induces $f : X \times I \to Y \times I$ by $(x, t) \mapsto (f(x), t)$. This induces $Cf : C(X) \to C(Y)$ and $Sf : S(X) \to S(Y)$.

Examples

$f : S^n \to S^n$ induces a map $Sf : S^{n+1} \to S^{n+1}$. $f : S^1 \to S^1$ by $z \mapsto z^2$ induces $Sf : S^2 \to S^2$

IMAGE 4

Proposition

$\deg(Sf) = \deg(f)$.

Proof

Consider the pair $(C(S^n), S^n \times \{0\})$ which induces

$$\overbrace{\tilde{H}_{n+1}(S^N)}^{=0} \longrightarrow \overbrace{\tilde{H}_{n+1}(C(S^n))}^{=0} \longrightarrow H_{n+1}(C(S^n), S^n \times \{0\}) \xrightarrow{\delta} \overbrace{\tilde{H}_n(S^n)}^{=\mathbb{Z}} \longrightarrow \overbrace{\tilde{H}_n(C(S^n))}^{=0} \longrightarrow$$

Hence $\mathbb{Z} \cong H_{n+1}(C(S^n), S^n \times \{0\}) \cong \tilde{H}_{n+1}(S(S^n)) = \tilde{H}_{n+1}(S^{n+1})$. Therefore

$$\begin{array}{ccccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \xrightarrow{\delta} & \tilde{H}_n(S^N) \\ \big\downarrow{\scriptstyle (Sf)_*} & & \big\downarrow{\scriptstyle (Cf)_*} & & \big\downarrow{\scriptstyle f_*} \\ H_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \longrightarrow & \tilde{H}_n(S^n) \end{array}$$

So $\deg(Sf) = \deg(f)$.

Remark

For any $k, n \in \mathbb{Z}_+$, by iterated suspension of the map $z \mapsto z^k$, we can construct $f : S^n \to S^n$ of degree k.

Remark

$Sf : S^{n+1} \to S^{n+1}$, pick $p \in S^{n+1}$ a pole, then $(Sf)^{-1}(p) = \{p\}$.

IMAGE 5

Hence $\deg(Sf|_p) = \deg(Sf) = k$.

May 12, 2025

Recall

Let X be a CW-Complex of finite dimension $X = X^0 \cup X^1 \cup \cdots \cup X^{\dim X}$.
X^0 is a discrete set of points.
X^1 is a gluing of $\{e^1_\alpha\}_{\alpha \in A}$ to X^0, where $e^1 = [-1, 1]$, by the attaching map $\varphi_\alpha : \partial e^1_\alpha \to X^0$.
X^{k+1} is the gluing of $\{e^{k+1}_\alpha\}_{\alpha \in A}$, where $e^{k+1} \cong D^{k+1}$, by $\varphi_\alpha : \partial e^{k+1}_\alpha \cong S^k \to X^k$.

Lemma

(a)

Let X be a CW-Complex of $\dim X$. Then

$$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \text{free abelian with a basis in 1-1 correspondence to} \{n\text{-cells}\} & \end{cases}$$

Proof

(X^n, X^{n-1}) is a good pair. So

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1}) = \tilde{H}_k\left(\bigvee_\alpha S^n_\alpha\right) = \bigoplus_\alpha \tilde{H}_k(S^n_\alpha).$$

If $k \neq n$, then $\tilde{H}_k(S^n_\alpha) = 0$.
If $k = n$, then $\tilde{H}_k(S^n_2) = \mathbb{Z}$ and $H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z}$.

(b)

$H_k(X^n) = 0$ if $k > n$.

Proof

The pair (X^n, X^{n-1}) gives a long exact sequence.

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \xrightarrow{\delta} H_k(X^{n-1}) \longrightarrow H_k(X^n)$$

$$\longrightarrow H_k(X^n, X^{n-1}) \xrightarrow{\delta} \cdots$$

Supposing both $k \neq n$ and $k+1 \neq n$, the first and last terms are zero and $H_k(X^{n-1}) \cong H_k(X^n)$. Then

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$

(c)

$i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Proof

If $k < n$, then

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^{\dim X}) = H_k(X)$$

Chain Complexes

$$
\begin{tikzcd}
 & & & \overbrace{H_n(X^{n+1},X^n)}^{=0} & \\
0 \arrow[dr] & & \overbrace{H_{n+1}(X^{n+1})}^{\cong H_n(X)} \arrow[ur] & & \\
 & H_{n+1}(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & & \\
H_{n+1}(X^{n+1},X^n) \arrow[ur, "\delta_n"] \arrow[rr, dashed, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, dashed, "d_n"] \arrow[dr, "\delta_{n+1}"] & & H_{n-1}(X^{n-1},X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j"] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd}
$$

This give a cellular chain complex $\{H_n(X^n,X^{n-1}),d_n\}$ with $d_n \circ d_{n+1} = 0$ because $\xrightarrow{j} \cdot \xrightarrow{\delta} = 0$. This defines a cellular homology $H_k^{CW}(X)$. We claim that $H_n^{CW}(X) \cong H_n(X)$.

Proof

$$
\begin{aligned}
H_n(X) &\cong H_n(X^{n+1} \\
&\cong H_n(X^n)/\ker i && \text{because } i \text{ is surjective} \\
&= H_n(X^n)/\operatorname{im}\delta_{n+1} && \text{because } \xrightarrow{\delta_{n+1}} \cdot \xrightarrow{i} \text{ is exact} \\
&\cong j(H_n(X^n))/j(\operatorname{im}\delta_{n+1}) && \text{because } j \text{ is injective} \\
&= \ker(\delta_n)/\operatorname{im}(d_{n+1}) && j(\operatorname{im}\delta_{n+1}) = \operatorname{im}(j\circ\delta_{n+1}) = \operatorname{im}(d_{n+1}) \\
&= \ker(d_n)/\operatorname{im}(d_{n+1}) && \ker(\delta_n) = \ker(j\circ\delta_n) = \ker d_n \\
&= H_n^{CW}(X)
\end{aligned}
$$

Applications

For

$$\cdots \longrightarrow H_n(X^n,X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1},X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$

where $H_n(X^n,X^{n-1}) = \bigoplus_\alpha \mathbb{Z}$

(1)

If a CW-Complex does not have any n-cells, then $H_n(X^n,X^{n-1}) = 0$ and $H_n(X) \cong H_n^{CW}(X) = 0$.

(2)

If a CW-Complex X has k-many n-cells, then $H_n(X^n,X^{n-1}) \cong \mathbb{Z}^k$. Then $H_n(X) \cong H_n^{CW}(X) = ker d_n/\operatorname{im} d_{n-1}$. $\ker d_n \leq H_n(X^n,X^{n-1}) = \mathbb{Z}^k$. Hence $\ker d_n$ and $H_n(X)$ can be generated by at most k many elements.

(3)

If X and Y are CW-complexes with $\{\varphi_\alpha : e_\alpha^n \to X^{n-1}\}$ and $\{\psi_\beta : e_\beta^n \to Y^{n-1}\}$ respectively, then $X \times Y$ has $\{\varphi_\alpha \times \psi_\beta : e_\alpha^m \times e_\beta^n \to (X \times Y)^{m+n-1}\}$ where $e_\alpha^m \times e_\beta^n \cong e^{m+n}$.
Consider $S^n \times S^n$ (for $n \geq 2$) where S^n is constructed by one 0-cell and one n-cell. Then $S^n \times S^n$ has one 0-cell ($\mathbb{Z}^1$), two n-cells ($\mathbb{Z}^2$) and one $2n$-cell ($\mathbb{Z}^1$).

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

so

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}^2 & k = n \\ 0 & \text{otherwise} \end{cases}.$$

(4)

Take $\mathbb{CP}^n$ as $\mathbb{C}^{n+1}/\sim$ or as $S^{2n+1}/\sim$ where $v \sim \lambda v$ and $\lambda v = (e^{i\theta} z_1, \dots, e^{i\theta} z_{n+1})$. Consider the set of vectors in S^{2n+1} whose last component is real and nonnegative. $D_+^{2n} = \{(w, \sqrt{1-|w|^2}) \in C^{n+1} \ : \ w \in C^n, \ |w| \leq 1\}$ is the graph of the function $w \mapsto \sqrt{1-|w|^2}$ defined on $\{w \ : \ |w| \leq 1\} \subseteq C^n$. So D_+^{2n} is homeomorphic to a disk $\{|w| \leq 1\} = D^{2n} \subseteq \mathbb{C}^n$. For any vector $v \in S^{2n+1}$, $v = (z_1, \dots, z_{n+1})$ if $z_{n+1} \neq 0$, then v is equivalent to a unique vector in D_+^{2n}. If $z_{n+1} = 0$, $\{(z_1, \dots, z_n, 0) \in S^{2n-1} \times \{0\}\} = S^{2n-1}$. So $q : S^{2n+1} \to \mathbb{CP}^n$ has that $q|_{D_+^{2n}}$ is a homeomorphism. Then $S^{2n-1}/\sim$ is exactly $\mathbb{CP}^{n-1}$. Therefore, we may view $\mathbb{CP}^n$ as gluing e^{2n} to $\mathbb{CP}^{n-1}$ by the attaching map $\partial e^{2n} = S^{2n-1} \to \mathbb{CP}^{n-1}$.
So $\mathbb{CP}^n$ has cells e^0, e^2, ..., e^{2n} and the cellular chain complex is

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

Recall that $\mathbb{RP}^n$ by $\S^n/\sim$ with $S^n \subseteq \mathbb{R}^{n+1}$ and $v \sim -v$, we may take the upper hemisphere D_+^n. For every $v \in S^n = (x_1, \dots, x_n)$, if $x_{n+1} \neq 0$ then v is equivaent to a unique vector in D_+^n where $q|_{D_+^n} : D_+^n \to \mathbb{RP}^n$ homemorphic to its image. If $x_{n+1} = 0$, then $\{(x_1, \dots, x_n, 0) \in S^n\}/\sim$ and $\mathbb{RP}^n$ is gluing e^n to $\mathbb{RP}^{n-1}$ via the attaching map $\varphi : \partial e^n = S^{n-1} \to \mathbb{RP}^{n-1}$ as the quotient map.

Computation

We want $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. For $n = 1$ we have

$$\begin{tikzcd}
0 \arrow[dr] & & & \\
& H_0(X^0) \arrow[dr, "j"] & & \\
H_1(X^1, X^0) \arrow[ur, "\delta"] \arrow[rr, "d_1"] & & H_0(X^0) \arrow[r] \arrow[dr] & 0 \\
& & & 0
\end{tikzcd}$$

where $d_1 = \delta : H_1(X^1, X^0) \to H_0(X^0)$. If X is connected, and $X^0 = \{v\}$, then $H_0(X^0) = \mathbb{Z}$ and $H_0(X) = H_0(X^0)/\operatorname{im} d_1$ implies that $\operatorname{im} d_1 = 0$.
For $n \geq 2$, $H_n(X^n, X^{n-1})$ is $\bigoplus_\alpha \mathbb{Z}$ and the generators are in one-to-one correspondence with $\{e_\alpha^n\}_\alpha$. We have a cellular boundary formula

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

where $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ and $\Delta_{\alpha\beta} : S^{n-1} = \partial e_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta} S_\beta^{n-1}$. $q_\beta : X^{n-1} \to S_\beta^{n-1}$ is obtained by collapsing everything in X^{n-1} except $(e_\beta^{n-1})^\circ$. For every n-cel e_α^n and every $(n-1)$-cell e_β^{n-1}, we obtain $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$.

Example

Suppose we have M_g, an orientable surface of genus g. M_g has one 0-cell, $2g$ 1-cells $(a_1, b_1, \ldots, a_g, b_{g\_}$ and one 2-cell. Then $d_1 = 0$, and $d_2(e_2)$ comes from $\Delta_{\alpha\beta} : S^2 = \partial e^2 \xrightarrow{\alpha} X^1 = \bigvee S^1 \xrightarrow{q_\beta} S_\beta^1$ which glues S^1 to S^1 by $a \cdot a^{-1}$. So $\deg(\Delta_{\alpha\beta}) = 0$ and $d_2(e_2) = 0$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so $H_2 = \mathbb{Z}$, $H_1 = \mathbb{Z}^{2g}$ and $H_0 = \mathbb{Z}$.

Example

N_g is a non-orientable surface of genus g. N_g has one 0-cell, g 1-cells $(a_1^2 a_2^2 \cdots a_g^2)$, and one 2-cell. We know that $d_1 = 0$. Consider $\Delta_{\alpha\beta} : S_\alpha^1 \to X^1 \to S_\beta^1$ which glues S^1 to S^1 by a^2 (i.e. $z \mapsto z^2$) and $\deg(\Delta_{\alpha\beta}) = 2$. So $d_2(e_2) = \sum_\beta 2e_\beta^1 = (2, 2, \ldots, 2) \in \mathbb{Z}^g$ and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^g/\operatorname{im} d_2 = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ and $H_2 = \ker d_2/0 = 0$.

May 14, 2025

Remaining Homology Topics

1. More examples of H_*^{CW}.

2. Euler characteristic.

3. Homology with coefficients.

4. Mayer-Vietoris Sequence.

5. $\pi_1/[\pi_1, \pi_1] = H_1$.

Recall: Cellular Chain Complex

$(H_n(X^n, X^{n-1}), d_n)$ where $H_n(X^n, X^{n-1})$ has a basis in one-to-one correspondence with the n-cells in X and $d_n(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$ with $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ for $\Delta_{\alpha\beta} : \partial e_n^\alpha = S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta} S_\beta^{n-1}$ where q_β collapses everything in X^{n-1} except

$\mathrm{int}(e_\beta^{n-1})$ to a point.

Example

$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$ with the attaching map $S^{k-1} = \partial e^k \to \mathbb{RP}^{k-1} = X^{n-1}$ as the quotient map (2-sheet covering).

IMAGE 1

So the cellular chain complex is

$0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$ with $d_n(e^n) = \deg(\Delta) e^{n-1}$ for $\Delta : S^{n-1} \xrightarrow{\varphi} X^{n-1} = \mathbb{RP}^{n-1} \xrightarrow{q} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$. Then Δ restricted to the upper (or lower) hemisphere is a homeomorphism to $S^{n-1} - \{\text{pt}\}$. So antipodal open sets differe only be the antipodal map

IMAGE 2

So $\deg(\Delta) = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$. If n is even, the chain complex is

$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$ which means $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$. Similarly for odd n,

$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$

so $H_n(\mathbb{RP}^n) = \mathbb{Z}$ and we conclude

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \text{ odd} \\ \mathbb{Z}_2 & 0 < k < n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Example

Recall for the torus $\mathbb{T}^2$, $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$, the degree is zero (see above).
For $\mathbb{T}^3$,

IMAGE 3

we have one 0-cell, three 1-cells, three 2-cells and one 3-cell and a chain complex

$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$ So $d_3(e^3) = \sum_\beta \deg(\Delta_{\alpha\beta}) e_\beta^2$ and $\Delta_{\alpha\beta} : \partial e^3 = S^2 \xrightarrow{\varphi} X^2 \xrightarrow{q_\beta} S_\beta^2$.

IMAGE 4

$$H_k(T^3) = \begin{cases} \mathbb{Z}^3 & k = 1,2 \\ \mathbb{Z} & k = 0,3 \\ 0 & \text{otherwise} \end{cases}.$$

Example

Consider $K \times S^1$ where K is the Klein bottle.

IMAGE 5

So $d_2(A) = 2b$, $d_2(B) = 0$ and $d_2(C) = 0$. However, for d_3 the front face changes by reflection in the back

IMAGE 6

So $d_3(e^3) = 2C$ and the degree is $1 + (-1) \cdot (-1) = 2$ and

$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$ We have that $\ker d_3 = 0$, $H_3 = 0$; $\ker d_2 = \langle B, C\rangle$, $\operatorname{im} d_3 = \langle 2C\rangle$, $H_2 = \ker d_2 / \operatorname{im} d_3 = \mathbb{Z} \oplus \mathbb{Z}_2$; $\ker d_1 = \langle a, b, c\rangle$, $\operatorname{im} d_2 = \langle 2b\rangle$, $H_1 = \mathbb{Z}^2 \oplus \mathbb{Z}_2$.

Homology with Coefficients (Moore Space)

Let G be a finitely generated abelian group and $n \in \mathbb{N}$ with $n \geq 1$. We can construct a CW-complex X such that

$$\tilde{H}_k(X) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}.$$

First, let us consider $G = 2m$. We start with $X^n = S^n$ (1 0-cell, 1 n-cell). Then construct $X = X^{n+1}$ by gluing in a $(n+1)$-cell e^{n+1} via the attaching map $\varphi : S^n = \partial e^{n+1} \to X^{n+1} = S^n$. Then $d_{n+1}(e^{n+1}) = \deg(\Delta) e^n$ where $\Delta : S^n = \partial e^{n+1} \to X^n = S^n$ is φ. If $\varphi : S^n \to S^n$ has degree m, then d_{n+1} is multiplication by m.

$0 \longrightarrow \mathbb{Z} \xrightarrow{d_{m+1}} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$ So $H_n(X) = \ker / \operatorname{im} = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m = G$. In general, $G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ a k-term sum. Use $X^n = \bigvee_{i=1}^k S^n$. For each $\mathbb{Z}_{m_i}$-factor, glue in e_α^{n+1} by $\varphi : \partial e_\alpha^{n+1} \to S_i^k$.
Consider X with ℓ-many $(n+1)$-cells, k-many n-cells and a single 0-cell for $\ell \leq k$. Then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^\ell & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0 \\ & & (1, 0, \cdots, 0) & \longmapsto & (m_i, 0, \ldots) & & \end{array}$$

So $\operatorname{im} d_{n+1} = m_1\mathbb{Z} \oplus \cdots \oplus m_\ell\mathbb{Z} \oplus 0 \subseteq \mathbb{Z}^k$.
Let G be an abelian group ($G = \mathbb{Z}$ or $G = \mathbb{Z}_m$). Then $C_n(X; G) = \left\{\sum_i n_i\sigma_i \ :\ \text{finite formal sums of } \sigma_i : \Delta^n \to X \text{ with } n_i \in G\right\}$.
We can similarly define $H_n(X, G)$, $\tilde{H}_n(X; G)$, $H_n(X, A; G)$, $H_n^{CW}(X; G)$, etc.

If we use $\mathbb{Z}_m = G$ as the coefficient with $\tilde{H}_k(X) = \begin{cases} \mathbb{Z}_m & k = m \\ 0 & k \neq m \end{cases}$,

$0 \longrightarrow \mathbb{Z}_m \xrightarrow{\cdot m} \mathbb{Z}_m \longrightarrow 0$ Consider the map $f : X \to X/S^n = S^{n+1}$. This induces $f_* : H_k(X) \to H_k(S^{n+1})$ which is $f_* = 0$ on all $H_k(X)$. If we use coefficients $\mathbb{Z}_m$ instead, we still induce $f_* : H_k(X; \mathbb{Z}_m) \to H_k(S^{n+1}; \mathbb{Z}_m)$, $H^{n+1}(X; \mathbb{Z}_m) = \mathbb{Z}_m$ and $H^n(X; \mathbb{Z}_m) = \mathbb{Z}_m$. For $k = n+1$, $f_* : \mathbb{Z}_m \to \mathbb{Z}_m$ is the identity map which implies that f is not null-homotopic.

Lemma

If $f : S^k \to S^k$ is of degree m, then $f_* : H_k(S^k; G) \to H_k(S^k; G)$ by $g \mapsto mg$. A homomorphism $\varphi : G_1 \to G_2$ induces $\varphi_\sharp : C_n(X; G_1) \to C_n(X, G_2)$ by $\sum n_i\sigma_i \mapsto \sum \varphi(n_i)\sigma_i$. Then $\partial\varphi_\sharp = \varphi_\sharp\partial$ and $\varphi_* : H_n(X; G_1) \to H_n(X; G_2)$. If $f : S^k \to S^k$

is of degree k, fix any $g \in G$ and set $\varphi : \mathbb{Z} \to G$ by $1 \mapsto g$.

$$\begin{array}{ccc} H_k(S^k;\mathbb{Z}) & \xrightarrow{f_*} & H_k(S^k;\mathbb{Z}) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ H_k(S^k;G) & \xrightarrow{f_*} & H_k(S^k;G) \end{array}$$

Euler Characteristic

Let X be a finite CW-complex with c_i the number of i-cells. Then $\chi(X) = \sum_{i=0}^{n}(-1)^i c_i$ with $n = \dim X$. For example, $\chi(S^2) = 1 + 1 = 2$, $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$, $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$, and $\chi(\mathbb{CP}^n) = 1 + 1 + \cdots + 1 = n$.

Theorem

$\chi(X) = \sum_{i=0}^{n}(-1)^i \operatorname{rank}(H_i(X))$.

Proof

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

So $Z_i = \ker d_i$, $B_i = \operatorname{im} d_{i+1}$ and $H_i = Z_i/B_i$. Then

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0$$

$$0 \longrightarrow Z_i = \ker d_i \longrightarrow C_i \longrightarrow B_{i-1} = \operatorname{im} d_i \longrightarrow 0$$

So $\operatorname{rank} Z_i = \operatorname{rank} B_i + \operatorname{rank} H_i$ and $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1} = \operatorname{rank} B_i + \operatorname{rank} H_i + \operatorname{rank} B_{i-1}$. Therefore

$$\sum (-1)^i \overbrace{\operatorname{rank} C_i}^{=c_i} = \sum (-1)^i \operatorname{rank} H_i.$$

So $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$.

May 19, 2025

Mayer-Vietoris Sequences

Given a space X and open sets $A, B \subseteq X$ such that $A \cup B = X$ (i.e. $\mathcal{U} = \{A, B\}$ is an open cover of X), we have the chain

$$C_n^{\mathcal{U}}(X) = \left\{ \sum_i n_i \sigma_i \,:\, \sigma_i : \Delta^n \to \text{ some } U \in \mathcal{U} \right\}$$

Fact: $H_n^{\mathcal{U}}(X) \cong H_n(X)$.
For convenience, we write $C_n^{\mathcal{U}}(X) = C_n(A + B)$ which induces a short exact sequence

$$0 \longrightarrow C_N(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \longrightarrow 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$(\alpha, \beta) \longmapsto \alpha + \beta$$

Verifying Exactness

φ is injective, since if $0 = \varphi(\alpha) = (\alpha, -\alpha)$, then $\alpha = 0$.
ψ is surjective, since any element in $C_n(A+B)$ and for $\sigma_i : \Delta^n \to A$ or B with $\alpha_i : \Delta^n \to A$ and $\beta_j : \Delta^n \to B$, we have

$$\sum_i n_i\sigma_i = \sum_i n_i\alpha_i + \sum_i n_i\beta_i = \psi\left(\sum_i n_i\alpha_i, \sum_i n_i\beta_i\right)$$

$\operatorname{im}\varphi \subseteq \ker\psi$ since $\psi(\varphi(\alpha)) = \psi(\alpha, -\alpha) = 0$
$\ker\psi \subseteq \operatorname{im}\varphi$, since if we suppose $(\alpha, \beta) \in C_n(A) \oplus C_n(B)$ such that $\psi(\alpha, \beta) = 0$. Then $\alpha \in C_n(A \cap B)$ and $\varphi(\alpha) = (\alpha, -\alpha) = (\alpha, \beta)$ (i.e. $(\alpha, \beta) \in \operatorname{im}\varphi$).

Long Exact Sequence

The short exact sequence induces

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\varphi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(X)$$

$$\xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

Example: Klein Bottle

The Klein bottle is created by gluing two Mobius strips along their boundary

IMAGE 1

Let A, B be two Mobius strips as subsets of the Klein bottle. Let U, V be neighborhoods of A and B respectively that deformation retract to A and B. Then $A \cap B$ is homotopic to the circle. Then

$$\overbrace{H_2(A \cap B)}^{=0} \longrightarrow \overbrace{H_2(A) \oplus H_2(B)}^{=0} \longrightarrow H_2(X)$$

$$\longrightarrow H_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(X)$$

$$\longrightarrow \overbrace{\tilde{H}_0(A \cap B)}^{=0}$$

Which leads to

$$0 \longrightarrow H_2(X) \longrightarrow H_1(A \cap B) \xrightarrow{\varphi_*} H_1(A) \oplus H_1(B) \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$0 \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{\varphi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$1 \longmapsto (2, -2)$$

In particular, ψ_* is injective. Hence $H_2(X) \to \mathbb{Z}$ is the zero map and $H_2(X) = 0$. Then also

$$H_1(X) = (\mathbb{Z} \oplus \mathbb{Z})/\ker\psi_* = (\mathbb{Z} \oplus \mathbb{Z})/\operatorname{im}\varphi_* = \langle a, b\rangle \,/\, \langle 2a - 2b\rangle = \langle a, a - b\rangle \,/\, \langle 2a - 2b\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Definition: Commutator

Let G be a group. $[G, G]$ is the normal subgroup generated by elements of the form $[g, h] = g^{-1}h^{-1}gh$. Then $G/[G, G]$ is Abelian.

Commutator of the Fundamental Group

$H_1 = \pi_1/[\pi_1, \pi_1]$.
We define a map $h : \pi_1(X, x_0) \to H_1(X)$. Let $f : \Delta^1 = [0, 1] \to X$ be a loop at x_0, such that f is also a singular 1-simplex (i.e. a cycle). So $\partial f = f(1) - f(0) = x_0 - x_0 = 0$. Hence we may assign $[f] \in H_1(X)$ to the loop f (i.e. $h(f) = [f]$). Write $f \simeq g$ for path homotopy and $f \sim g$ when $f - g$ is a boundary.
h is well defined, since

1. if f is homotopic to a constant loop, then $f \sim 0$ (i.e. f is a boundary). We treat $H : D^2 \to X$ as a singular 2-simplex, call it σ. Then $\partial\sigma = f$.

2. if $f \simeq g$, then $f \sim g$. Let H be a path homotopy between f and g.

IMAGE 2

Then $\partial L = d - g - Cx_0$, $\partial R = d - Cx_0 - f$ and $\partial(L - R) = f - g$. This shows that $h : \pi_1(X, x_0) \to H_1(X)$ is well-defined.
h is a group homomorphism. We need to show that $h(f \cdot g) = h(f) + h(g)$ (i.e. $f \cdot g \sim f + g$).

IMAGE 3

So $\sigma : \Delta^2 \to X$ defined by the filling in of the 2-simplex has $\partial\sigma = f \cdot g - f - g$.
h is surjective. For $\sigma_i : \Delta^i \to X$, let $\sum_i n_i\sigma_i \in C_1(X)$ be a 1-cycle. Then

$$0 = \partial\left(\sum_i n_i\sigma_i\right) = \sum_i n_i\left(\sigma_i(1) - \sigma_i(0)\right).$$

Let S be the set of distinct points in the list $\{\sigma_i(0), \sigma_i(1) \,:\, |i = 1, \ldots, k\}$ for $m_p \in \mathbb{Z}$. Then

$$\sum_i n_i(\sigma_i(1) - \sigma_i(0)) = \sum_{p \in S} m_p \cdot p.$$

and $m_p = 0$ for all $p \in S$. For each σ_i, we consider a loop η_i at x_0 by

IMAGE 4

For any $p \in S$, β_p is a path from x_0 to p. Then $h(\eta_i) = \beta_{\sigma_i(0)} + \sigma_i - \beta_{\sigma_i(1)}$. Now consider a loop $\eta_1^{n_1} \cdots \eta_k^{n_k}$ at x_0.

$$h(\eta_1^{n_1} \cdots \eta_k^{n_k}) = \sum_i n_i h(\eta_i) = \sum_i n_i\sigma_i + \sum_i n_i(\beta_{\sigma_1(0)} - \beta_{\sigma_1(1)}) = \sum_i n_i\sigma_i + 0$$

$\ker(h) = [\pi_1, \pi_1]$. Since $[\pi_1, \pi_1]$ is generated by $f^{-1}g^{-1}fg$, $h(f^{-1}g^{-1}fg) = -h(f) - h(g) + h(f) + h(g) = 0$. So $[\pi_1, \pi_1] \subseteq \ker h$.

Lemma:

Let G be a group, and let w be a word in G. Suppose that for each $g \in G$ its exponent in w adds up to zero, then $w \in [G,G]$.

Proof

Let $\pi : G \to G/[G,G]$ be the quotient map and $\pi(w) = 0$ (i.e. $w \in \ker \pi = [G,G]$ because if $w = g^{k_1} \cdots g^{k_2} \cdots g^{k_3} \cdots$ then $\pi(w) = \pi(g)^{k_1+k_2+k_3+\cdots} = 0$.

Commutator of the Fundamental Group Continued

Let f be a loop at x_0 such that $h(f) = 0$ (i.e. as a singular 1-simplex). f is a boundary. Hence there is $\sum_i n_i\sigma_i \in C_2(X)$ such that ($\sigma_i : \Delta^2 \to X$)

IMAGE 5

$$f = \partial\left(\sum_i n_i\sigma_i\right) = \sum n_i(\alpha_i + \beta_i + \gamma_i)$$

Let S be the set of disjoint edges in the list $\{\alpha_i, \beta_i, \gamma_i \, : \, i = 1, \ldots, k\}$. Then

$$f = \sum n_i(\alpha_i + \beta_i + \gamma_i) = \sum_{e \in S} m_e \cdot e$$

in $C_1(X)$. Hence $m_e = 1$ when $e = f$ and $m_e = 0$ otherwise. For each σ_i, we draw a loop η_i at x_0 by joining

IMAGE 6

Then each η_i is homotopic to the boundary of σ_i (i.e. null-homotopic). Let us consider a loop $\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1} \simeq f^{-1}$. Each η_i is a product of 3 loops. Hence $\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1}$ is a word in π_1 in $S' \cup \{f\}$ where S' is the collection of all the generated loops with basepoint at x_0. The exponent of such a loop in this word is either m_e when $e \neq f$ or $m_e - 1$ when $e = f$. So it is always zero by the precedeing. This shows that

$$[f^{-1}] = [\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1}] \in [\pi_1, \pi_1].$$

Mayer-Vietoris for Reduced Homology

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & \\
& C_0(A \cap B) \arrow[r, "\epsilon"] \arrow[d, "\varphi_*"] & \mathbb{Z} \arrow[r] \arrow[d, "\varphi"] & 0 \\
& C_0(A) \oplus C_0(B) \arrow[r, "\epsilon\oplus\epsilon"] \arrow[d, "\psi_*"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "\psi"] & 0 \\
& C_0(A+B) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 &
\end{tikzcd}$$

Where $\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum_i n_i\sigma_i \mapsto \sum n_i$ is surjective. Then $\alpha \in \sum_i n_i\sigma_i$, and we have $\varphi\left(\sum_i n_i\right)$ and $(\epsilon \oplus \epsilon)\left(\sum_i n_i\sigma_i, -\sum_i n_i\sigma_i\right) = \left(\sum_i n_i, -\sum_i n_i\right)$. So assign $\varphi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ by $1 \mapsto (1, -1)$ and let $\left(\sum_i n_i\alpha_i, \sum_i m_i\beta_i\right) \in C_0(A) \oplus C_0(B)$. Then we have that $\epsilon\left(\sum_i n_i\alpha_i, \sum_i m_i\beta_i\right) = \sum_i n_i + \sum_i m_i$ and $\psi\left(\sum_i n_i, \sum_i m_i\right)$ is assigned to $\psi_i : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $(m,n) \mapsto m+n$. Then the above diagram commutes.

May 21, 2025

Recall: de Rahm Cohomology

If M^n is a continuous manifold, $\Omega^p(M)$ is the collection of k-forms on M and $d : \Omega^p(M) \to \Omega^{p+1}(M)$ with $d \circ d = 0$. This gives a cochain

$$\cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots$$

It defines the de Rahm cohomology $H^p_{\mathsf{dR}}(M) = \ker d_p / \operatorname{im} d_{p-1}$.
Our goal is the de Rahm Theorem:

$$H^p_{\mathsf{dR}}(M) \cong H^p(M;\mathbb{R})(= \operatorname{Hom}(H_p(M),\mathbb{R})).$$

For $p = 1$ we can construct a map $I : H^1_{\mathsf{dR}}(M) \to \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$ by $[\omega] \mapsto I[\omega] : \pi_1(M,x) \to \mathbb{R}$ where $(I[\omega])[\gamma] := \int_{\tilde{\gamma}} \omega$. TO be precise, we pick a piecewise smooth $\tilde{\gamma}$ at x_0 such that $[\tilde{\gamma}] = [\gamma]$ (well-defined because for α, β piecewise smooth and "smoothly homotopic", $\int_\alpha \omega = \int_\beta \omega$).
Then I is well defined because it is independent of the choice of $\tilde{\gamma}$ and ω. In the latter case, if $\omega = \omega' + df$ for $f \in \Omega^0(M)$, then

$$\int_{\tilde{\gamma}} \omega - \omega' = \int_{\tilde{\gamma}} df = f(\tilde{\gamma}(1)) - f(\tilde{\gamma}(0)) = 0.$$

I is injective, because if we suppose $I[\omega] = 0 \in \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$, then $\int_{\tilde{\gamma}} \omega = I[\omega][\gamma] = 0$ for all $\gamma \in \pi_1(M,x)$. Then we claim that ω is conservative, because for any piecewise smooth loop α we can let y be a point on α and draw a composed curve with β from x to y. Then $\beta\alpha\beta^{-1}$ is based at x and

$$0 = \int_{\beta\alpha\beta^{-1}} \omega = \int_\beta \omega + \int_\alpha \omega - \int_\beta \omega = \int_\alpha \omega$$

Then, recall that conservative implies exactness since $f(y) = \int_{\gamma_{xy}} \omega$ holds for γ_{xy} from x to y. Therefore $[\omega] = 0$.
I is surjective (to be proved later) since if $\gamma \in \pi_1(M,x)$ such that $\gamma^k = \mathrm{id}$, then for every $\varphi \in \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$ we have that $\varphi(\gamma) = 0$.

Corollary

If $\pi_1(M)$ is torsion (in particular if $\pi_1(M)$ is finite), then $H^1_{\mathsf{dR}}(M) = \{0\}$.

(A Little Bit of) Cohomology

In homology, we start with a chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

Let G be an abelian group ($G = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{Z}_m$), and define $C^*_n = \operatorname{Hom}(C_n, G)$ and $\partial^* : C^*_n \to C^*_{n+1}$ by $\varphi \mapsto \partial^*\varphi$ where $\partial^*\varphi$ is defined by $\varphi \circ \partial$. This gives a cochain

$$\cdots \longleftarrow C^*_{n+1} \xleftarrow[\partial^*]{} C^*_n \xleftarrow[\partial^*]{} C^*_{n-1} \longleftarrow \cdots$$

with $\partial^* \circ \partial^* = 0$.

Definition: Cohomology Group

The cohomology group $H^n(X,G)$ is $\ker\partial^*/\operatorname{im}\partial^*$.

Example

Given

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

we have a dual

$$0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \xleftarrow[\cdot 2]{} \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \longleftarrow 0$$

Then we have $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$, $H_2 = 0$ and $H_3 = \mathbb{Z}$ while $H^0 = \mathbb{Z}$, $H^1 = 0$, $H^2 = \mathbb{Z}_2$ and $H^3 = \mathbb{Z}$. In general, $H^n(X,G) \ncong \operatorname{Hom}(H_n(X),G)$.

Fact (1)

Let T_n be the torsion subgroup of H_n. Then $H^n(X,\mathbb{Z}) \cong T_{n-1} \oplus (H_n/T_n)$.

Fact (2)

$H^n(X,\mathbb{R}) \cong \operatorname{Hom}(H_n(X),\mathbb{R})$. For example, we can dual by $G = \mathbb{R}$

$$0 \longleftarrow \mathbb{R} \xleftarrow[0]{} \mathbb{R} \xleftarrow[\sim]{} \mathbb{R} \xleftarrow[0]{} \mathbb{R} \longleftarrow 0$$

with $\mathbb{R}$ coefficients. Then $H^0 = \mathbb{R}$, $H^1 = 0$, $H^2 = 0$ and $H^3 = \mathbb{R}$.

Integration

Take $H_p(M)$ and a singular n-simplex $\sigma : \Delta^p \to M$. We want $I : H^p_{\text{dR}}(M) \to \operatorname{Hom}(H_p(M),\mathbb{R})$ by $[\omega] \mapsto I[\omega]$ where $(I[\omega])[\sigma] = \int_\sigma \omega$. To make this work, we consider smooth σ and the collection $C_p^\infty(M) = \{\sigma : \Delta^p \to M \text{ smooth}\}$. This gives a chain complex

$$\cdots \longrightarrow C^\infty_{p+1} \longrightarrow C^\infty_p \longrightarrow C^\infty_{p-1} \longrightarrow \cdots$$

Then it has homology group $H_p^\infty(M)$.

Fact

The inclusion map $\iota : C_p^\infty(M) \to C_p(M)$ induces an isomorphism $\iota_* : H_p^\infty(M) \to H_p(M)$. Then we can consider instead $I : H_{\text{dR}^p(M)} \to \operatorname{Hom}(H_p^\infty(M),\mathbb{R})$. Then if $\sigma : \Delta^p \to M$ is smooth, we can write $\int_\sigma \omega := \int_{\Delta^p} \sigma^*\omega$ where $\Delta^p \subseteq \mathbb{R}^p$ is formed by vertices $[v_0,\ldots,v_p]$ of the form $v_0 = (0,\ldots,0)$ and $v_i = (0,\ldots,1,\ldots,0)$.

Stoke's Theorem (for Integration over Smooth Chains)

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

where $\partial\sigma = \sum_i (-1)^i \sigma|_{i\text{-th face}}$. More precisely, $F_i : \Delta^{p-1} \to i$-th face of Δ^p by $[v_0,\ldots,v_{p-1}] \mapsto [v_0,\ldots,\hat{v}_i,\ldots,v_p]$. So $\partial\sigma = \sum_i (-1)^i \sigma \circ F_i$. We need to check orientation, so write

$$\int_\sigma d\omega = \int_{\Delta^p} \sigma^*(d\omega) = \int_{\Delta^p} d(\sigma^*\omega) = \int_{\partial\Delta^p} \sigma^*\omega = \sum_i \int_{\partial_i\Delta^p} \sigma^*\omega$$

where $\partial\Delta^p$ has outward orientation and $\partial_i\Delta^p$ is the i-th face with outward orientation. On the right-hand side

$$\int_{\partial\sigma} \omega = \sum_i (-1)^i \int_{\sigma\circ F_i} \omega = \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^*\omega$$

where $\Delta^{p-1} \subseteq \mathbb{R}^p$ has a standard orientation given by $[e_1^{(p-1)}, \dots, e_{p-1}^{(p-1)}]$. So $F_i : \Delta^{p-1} \to \partial_i\Delta^p$, where the domain has standard orientation and the image has outward orientation. Then F_i maps $e_1^{(p-1)} \mapsto e_1^{(p)}, \dots, e_{i-1}^{(p-1)} \mapsto e_{i-1}^{(p)}$, $e_i^{(p-1)} \mapsto e_{i+1}^{(p)}, \dots, e_{p-1}^{(p-1)} \mapsto e_p^{(p)}$. Then

$$((-e_i) \lrcorner\, d\,\mathrm{vol}_p)[F_i(e_1^{(p-1)}, \dots, F_i(e_{p-1}^{(p-1)}] = d\,\mathrm{vol}_p[-e_i, e_1, \dots, \hat{e}_i, \dots, e_p] = (-1)^i \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^*\omega = \sum_i (-1)^{2i} \int_{\partial_i\Delta^p} \sigma^*\omega$$

Continuing...

For $I : H^p_{\mathrm{dR}}(M) \to \mathrm{Hom}(H_p(M), \mathbb{R})$ by $[\omega] \mapsto I[\omega]$ with $I[\omega][c] = \int_c \omega$, recall that $c = \sum_i n_i\sigma_i$ and $\sigma_i : \Delta^p \to M$ smooth. Then $\int_c \omega := \sum_i n_i \int_{\sigma_i} \omega$.

I is well defined, since if $\sigma' = \sigma + \partial\eta$ for $\eta : \Delta^{p-1} \to M$ smooth, $\int_{\sigma'} \omega = \int_\sigma \omega + \int_{\partial\eta} \omega$. But by Stokes', $\int_{\partial\eta} \omega = \int_\eta d\omega = 0$.
Secondly, if $\omega' = \omega + d\eta$ for $\eta \in \Omega^{p-1}(M)$, then $\int_\sigma \omega' = \int_\sigma \omega + \int_\sigma d\eta$ and again $\int_\sigma d\eta = \int_{\partial\sigma} \eta = 0$.

Naturality in de Rahm Cohomology

Given $F : M \to N$ smooth,

$$\begin{tikzcd}
H^p_{\mathrm{dR}}(N) \arrow[r, "F^*"] \arrow[d, "I"] & H^p_{\mathrm{dR}}(M) \arrow[d, "I"] \\
\mathrm{Hom}(H_p(N), \mathbb{R}) \arrow[r, "F^*"] & \mathrm{Hom}(H_p(M), \mathbb{R})
\end{tikzcd}$$

Let $[\omega] \in H^p_{\mathrm{dR}}(N)$ and $[0] \in H^p(M)$, then

$$(I \circ F^*[\omega])([0]) = (I \circ [F^*\omega])([0]) = \int_\sigma F^*\omega = \int_{F\circ\sigma} \omega = I[\omega](F \circ \sigma) = (F^* \circ I[\omega])(\sigma).$$

so the diagram commutes.

Mayer-Vietoris for de Rahm Cohomology

If $M = U \cup V$ for U, V open, then

$$\begin{tikzcd}
\cdots \arrow[r] & H^p_{\mathrm{dR}}(M) \arrow[r] \arrow[d, "I"] & H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \arrow[r] \arrow[d, "I\oplus I"] & H^p_{\mathrm{dR}}(U \cap V) \arrow[r, "\delta"] \arrow[d, "I"] & H^{p+1}_{\mathrm{dR}}(M) \arrow[r] \arrow[d, "I"] & \cdots \\
\cdots \arrow[r] & H^p_{\mathrm{dR}}(M, \mathbb{R}) \arrow[r] & H^p_{\mathrm{dR}}(U, \mathbb{R}) \oplus H^p_{\mathrm{dR}}(V, \mathbb{R}) \arrow[r] & H^p_{\mathrm{dR}}(U \cap V, \mathbb{R}) \arrow[r, "\delta"] & H^{p+1}(M, \mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

Recall that

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^*\oplus l^*} \Omega^k(U)\oplus\Omega^k(V) \xrightarrow{i^*-j^*} \Omega^k(U\cap V) \longrightarrow 0$$

$$\omega \longmapsto (\omega|_U, \omega|_V) \longrightarrow 0$$

$$(\omega,\eta) \longmapsto (\omega|_{U\cap V} - \eta|_{U\cap V})$$

So we have a short exact sequence

$$0 \longrightarrow C_p(U\cap V) \xrightarrow{\alpha} C_p(U)\oplus C_p(V) \xrightarrow{\beta} C_p(M) \longrightarrow 0$$

$$\sigma \longmapsto (\sigma, -\sigma)$$

$$(\omega,\eta) \longmapsto \sigma+\eta$$

which we dualize to

$$0 \longrightarrow C_p^*(M) \xrightarrow{\beta^*} C_p^*(U)\oplus C_p^*(V) \xrightarrow{\alpha^*} C_p^*(U\cap V) \longrightarrow 0$$

$$\varphi \longmapsto \beta^*\varphi$$

$$(\varphi,\psi) \longmapsto \alpha^*(\varphi,\psi)$$

So ultimately we have

$$\begin{tikzcd}
0 \arrow[r] & \Omega^p(M) \arrow[r] \arrow[d, "d"] & \Omega^p(U)\oplus\Omega^p(V) \arrow[r] \arrow[d, "d"] & \Omega^p(U\cap V) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{p+1}(M) \arrow[r] & \Omega^{p+1}(U)\oplus\Omega^{p+1}(V) \arrow[r] & \Omega^{p+1}(U\cap V) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & C_p^*(M) \arrow[r] \arrow[d, "d"] & C_p^*(U)\oplus C_p^*(V) \arrow[r] \arrow[d, "d"] & C_p^*(U\cap V) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C_{p+1}^*(M) \arrow[r] & C_{p+1}^*(U)\oplus C_{p+1}^*(V) \arrow[r] & C_{p+1}^*(U\cap V) \arrow[r] & 0
\end{tikzcd}$$

Where I maps from $\Omega^p(M)$ to $C_p^*(M)$ and the entire three dimensional diagram commutes.

May 28, 2025

Recall

If $M = U \cap V$, then we may dualize by $C^p = \mathrm{Hom}(C_p;\mathbb{R})$ and

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
C_p(U\cap V) \arrow[d, "\alpha"] & c \arrow[d, mapsto] & C^p(M) \arrow[d, "\beta^*"] & \Omega^p(M) \arrow[l, "I"'] \arrow[d] & \omega \arrow[d, mapsto] \\
C_p(U)\oplus C_p(V) \arrow[d, "\beta"] & \substack{(c,-c)\\(c,c')} \arrow[d, mapsto] & C^p(U)\oplus C^p(V) \arrow[d, "\alpha^*"] & \Omega^p(U)\oplus \Omega^p(V) \arrow[l, "I\oplus I"'] \arrow[d] & \substack{(\omega|_U,\omega|_V)\\(\omega,\omega')} \arrow[d, mapsto] \\
C_p(M) \arrow[d] & c+c' & C^p(U\cap V) & \Omega^p(U\cap V) \arrow[l, "I"'] \arrow[d] & \omega|_{U\cap V}-\omega'|_{U\cap V} \\
0 & & 0 & 0 &
\end{tikzcd}$$

where $(\beta^*\varphi)(c,c') = (\varphi\circ\beta)(c,c') = \varphi(c+c')$ and $(\alpha(\varphi,\varphi'))(c) = (\varphi,\varphi')(\alpha(c)) = (\varphi,\varphi')(c,-c) = \varphi(c)-\varphi'(c)$. $C^p(U)\oplus C^p(V)$ acts on $C_p(U)\oplus C_p(V)$ by $(\varphi,\varphi')(c,c') := \varphi(c)+\varphi'(c')$.
$I : \Omega^p(*) \to C^p(*)$ by $I(\omega)(c) := \int_c \omega$. We have that the following commutes

$$\begin{tikzcd}
& C^p(M) \arrow[r, "\partial^*"] & C^{p+1}(M) \\
\Omega^p(M) \arrow[ur, "I"] \arrow[r, "d"] & \Omega^{p+1}(M) \arrow[ur, "I"] &
\end{tikzcd}$$

since for $\omega \in \Omega^p(M)$ we have $(I(d\omega))(c) = \int_c d\omega = \int_{\partial c}\omega = I(\omega)(\partial c) = \partial^*(I(\omega))(c)$. Similarly, for $(\omega,\omega') \in \Omega^p(U)\oplus\Omega^p(V)$,

$$\begin{aligned}
(\alpha^*(I\oplus I)(\omega,\omega'))(c) &= (I\oplus I)(\omega,\omega')(c,-c) \\
&= (I(\omega), I(\omega'))(c,-c) \\
&= I(\omega)(c) + I(\omega')(-c) \\
&= \int_c \omega - \omega' \\
&= I(\omega|_{U\cap V} - \omega'|_{U\cap V})(c).
\end{aligned}$$

Then we may apply the five lemma to see that

$$\begin{tikzcd}
H^{p-1}_{\mathrm{dR}}(U)\oplus H^{p-1}_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & H^{p-1}_{\mathrm{dR}}(U\cap V) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(M) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(U)\oplus H^p_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(U\cap V) \arrow[d] \\
H^{p-1}(U)\oplus H^{p-1}(V) \arrow[r] & H^{p-1}(U\cap V) \arrow[r] & H^p(M) \arrow[r] & H^p(U)\oplus H^p(V) \arrow[r] & H^p(U\cap V)
\end{tikzcd}$$

Theorem (de Rahm Theorem)

$I : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$ is an isomorphism.

Proof

For convenience, we say a manifold M is de Rahm is $I : H^p_{\text{dR}}(M) \to H^p(M;\mathbb{R})$ is an isomorphism.

Step 1

A disjoint union of de Rahm manifolds is de Rahm. In fact, $M = \coprod_{\alpha\in A} M_\alpha$. We have shown that $H^p_{\text{dR}}(M) = \prod_{\alpha\in A} H^p_{\text{dR}}(M_\alpha)$ and can show the same for $H^p(M) = \prod_{\alpha\in A} H^p(M_\alpha)$. If $I : H^p_{\text{dR}}(M_\alpha) \to H^p(M_\alpha;\mathbb{R})$ is an isomorphism for each α, then $I : H^p_{\text{dR}}(M) \to H^p(M;\mathbb{R})$.

Step 2

Every convex open set in $\mathbb{R}^n$ is de Rahm. Let $U \subseteq \mathbb{R}^n$ convex. We know that

$$H^p_{\text{dR}}(U) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p \geq 1 \end{cases} \qquad\qquad H^p(U;\mathbb{R}) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p \geq 1 \end{cases}$$

and only need to check $p = 0$. We see that $H^0_{\text{dR}}(U) = \{f : U \to \mathbb{R} \,:\, df = 0\}$ are the constant functions and $C_0(U)$ is generated by $\sigma : \{p\} \to U$. For $I : H^0_{\text{dR}}(M) \to H^0(M;\mathbb{R})$, we know tha t$I$ is injective (and therefore an isomorphism) since if $f \in \ker I$ (i.e. $f(p) = I(f)(p) = 0,\ \forall p \in U$), then $f \equiv 0$ on U and $\ker I = \{0\}$.

Definition: de Rahm Cover / de Rahm Basis

We say that an open cover $\mathcal{U} = \{U_\alpha\}$ of M is de Rahm if every non-empty, finite intersection $\bigcap_{i=1}^k U_{\alpha_i}$ is de Rahm. If, in addition, $\mathcal{U}$ is a basis for M, then we say that $\mathcal{U}$ is a de Rahm basis.

Step 3

If M has a finite de Rahm cover, then M is de Rahm. Let $\{U_i\}_{i=1}^k$ be a de Rahm cover. We prove by induction on k. For $k = 2$, $M = U \cap V$, we conclude by Mayer-Vietoris sequences and the five lemma (above) that $I : H^p_{\text{dR}}(M) \to H^p(M;\mathbb{R})$. Now, supposing this holds for arbitrary k, we set $U = \bigcup_{j=1}^k U_j$ and $V = U_{k+1}$. Then U is de Rahm by inductive hypothesis, and again by Mayer-Vietoris and five lemma.

Remark

As a fact, if M^n is a closed manifold, then M has a finite good cover $\{U_i\}_{i=1}^k$ (i.e. every non-empty intersection is diffeomorphic to $\mathbb{R}^n$).

Step 4

If M has a de Rahm basis $\mathcal{U}$, then M is de Rahm.
Fact: M admits an exhaustion function $f : M \to \mathbb{R}$ such that $f^{-1}([0,a])$ is compact (i.e. f is proper) for all $a > 0$. For each $m \in \mathbb{Z}_+$ consider $A_m \in \{x \in M \,:\, m \leq f(x) \leq m+1\}$ and $A'_m = \{x \in M \,:\, m - 1/2 \leq f(x) \leq m + 3/2\}$. Then A_m and A'_m are compact and $A_m \subseteq A'_m$. For each $x \in A_m$, there is $U_x \in \mathcal{U}$ such that $x \in U_x \subseteq A'_m$ so $\{U_x \,:\, x \in A_m\}$ is an open cover of A_m and admits a finite subcover. Let B_m be the union of such a finite subcover. Then $A_m \subseteq B_m \subseteq A'_m$. Moreover, B_m fulfills the assumptions of Step 3 and is therefore de Rahm.
Note: if $m \neq \tilde{m}$, then B_m cannot intersect $B_{\tilde{m}}$ when $\tilde{m} \neq m-1, m+1$.
Set $U = \dot\bigcup_{m \text{ odd}} B_m$ and $V = \dot\bigcup_{m \text{ even}} B_m$ which are both de Rahm by step 1. We observe that $(U \cap V) = \dot\bigcup (B_m \cap B_{m+1})$ which are each de Rahm by step 3. This further implies that $U \cap V$ is de Rahm by step 1. Now, $M = U \cup V$ with $U, V, U \cap V$ de Rahm. By step 3 again, M is de Rahm.

Step 5

Every open set in $\mathbb{R}^n$ is de Rahm. This is because for $U \subseteq \mathbb{R}^n$ open, U has a basis $\mathcal{U}$ whose elements are Euclidean balls. Therefore $\mathcal{U}$ is de Rahm by step 2 and U is de Rahm by step 4.

Step 5

Every manifold is de Rahm, because they have a basis whose elements are diffeomorphic to open sets in $\mathbb{R}^n$.

Poincaré Duality (for de Rahm Cohomology)

Let M^n be a closed, orientable manifold. Define $P : \Omega^k(M) \to \Omega^{n-k}(M)^*$ by $(P(\omega))(\eta) = \int_M \omega \wedge \eta$ for $\omega \in \Omega^k$ and $\eta \in \Omega^{n-k}$. It induces $P : H^k_{\text{dR}}(M) \to H^{n-k}_{\text{dR}}(M)^*$ by $(P[\omega])[\eta] = \int_M \omega \wedge \eta$.

Then P is well defined, since we observe that for η closed $d(\alpha \wedge \eta) = d\alpha \wedge \eta \pm \overbrace{\alpha \wedge d\eta}^{=0}$ and

$$\int_M (\omega + d\alpha) \wedge \eta = \int_M \omega \wedge \eta + \int_M d\alpha \wedge \eta = \int_M \omega \wedge \eta + \int_M d(\alpha \wedge \eta) = \int_M \omega \wedge \eta + \overbrace{\int_{\partial M} \alpha \wedge \eta}^{=0}$$

Similarly, $\int_M \omega \wedge \eta = \int_M \omega \wedge (\eta + d\rho)$.
If, instead, M is an orientable manifold without boundary, then $P : H^k_{\text{dR}}(M) \to H^{n-k}_C(M)^*$ (i.e. compactly supported) by $(P[\omega])[\eta] = \int_M \omega \wedge \eta$.

Theorem (Poincaré)

If M^n is closed and orientable, then $H^k_{\text{dR}}(M) \cong H^{n-k}_{\text{dR}}(M)^* \cong H^k_{\text{dR}}(M)^{**}$ which implies that $\dim H^k_{\text{dR}}(M) < +\infty$ which implies that $\dim H^k_{\text{dR}}(M) = \dim H^{n-k}_{\text{dR}}(M)$. Also

$$\dim H^k_{\text{dR}}(M) = \dim H^k(M;\mathbb{R}) = \dim H_k(M;\mathbb{R}) = \operatorname{rank} H_k(M).$$

When n is odd, the Euler characteristic

$$\chi(M) = \sum_k (-1)^k \operatorname{rank} H_k(M) = \sum_k (-1)^k \dim H^k_{\text{dR}}(M) = 0.$$

Ingredients to Prove Poincaré Theorem

We say that a manifold M is Poincaré if $P : H^k_{\text{dR}}(M) \to H^{n-k}_C(M)^*$ is an isomorphism. Similarly to the proof of the de Rahm theorem, we can define a Poincaré cover and a Poincaré basis.

- Step 1: If $M = \coprod M_\alpha$ with each M_α Poincaré, then M is Poincaré.

- Step 2: If M is a convex open subset in $\mathbb{R}^n$, then M is Poincaré. If $M = U$ is a convex open subset of $\mathbb{R}^n$, we know that

$$H^{n-k}_C(U) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad\qquad H^k_{\text{dR}}(U) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

so we only need check $k = 0$. Then $P : H^0_{\text{dR}}(U) \to H^n_C(U)^*$ is given by $P(c)(\omega) = \int_U c\omega$ for $c : U \to \mathbb{R}$ constant and $\omega \in \Omega^n_C(U)$. If $P(c) = 0$, then $\int_U c\omega = 0$ for all $\omega \in \Omega^n_C(U)$. In particular, we can use a bump function to construct $\omega \in \Omega^n_C(U)$ such that $\int_U \omega = 1$ ($\omega = f\, dx^1 \wedge \cdots \wedge dx^n$). Then $c \int_U \omega = 0$ implies $c = 0$ and P is injective.

June 2nd, 2025

Unfortunately, I was absent for this lecture. I believe much of the content was the completion of the proof of Poincaré duality and the beginning of the proof of Kunneth's formula. The statement of the latter is below.

June 4th, 2025

Recall: Kunneth Formula

Given $M = U \cup V$ and $d : H^p_{\text{dR}}(U \cap V) \to H^{p+1}_{\text{dR}}(M)$ by $[\omega] \mapsto [\eta]$ where $[\eta]$ is defined as follows: let $\{\rho_U, \rho_V\}$ be a partition of unity with respect to the open cover $\{U, V\}$. Then either

$$\eta = \begin{cases} d(\rho_V \omega) & \text{on } U \\ -d(\rho_U \omega) & \text{on } V \end{cases}.$$

Note that η is supported on $U \cap V$.

Kunneth Formula: Finishing the Proof

We need to check that

$$\begin{array}{ccc} H^P(U \cap V) \otimes H^{n-p}(N) & \xrightarrow{d} & H^{p+1}(M) \otimes H^{n-p}(N) \\ \big\downarrow \Phi & & \big\downarrow \\ H^n((U \cap V) \times N) & \xrightarrow{\quad d \quad} & H^{n+1}(M \times N \end{array}$$

commutes where $\Phi(\omega \otimes \eta) = (\pi^* \omega) \wedge (\rho^* \eta)$ for $\pi : M \times N \to M$ and $\rho : M \times N \to N$. Let $[\omega] \otimes [\eta] \in H^p_{\text{dR}}(U \cap V) \otimes H^{n-p}_{\text{dR}}(N)$. Then

$$\Phi((d[\omega] \otimes \sigma) = [(\pi^* \eta) \wedge (\rho^* \sigma)] = \pi^*(d \rho_V \omega) \wedge (\rho^* \sigma).$$

Since $M \times N = (U \times N) \cup (V \times N)$, we can define $\pi^* \rho_U : M \times N \to \mathbb{R}$ by $\pi^* \rho_U(x, y) = \rho_U(x)$ and similarly $\pi^* \rho_V(x, y) := \rho_V(x)$. Then again $\{\pi^* \rho_U, \pi^* \rho_V\}$ is a partition of unity with respect to $\{U \times N, V \times N\}$. So we have

$$d[\Phi(\omega \otimes \sigma)] = d[(\pi^* \omega) \wedge (\rho^* \sigma)] = \begin{cases} d(\pi^* \rho_V((\pi^* \omega) \wedge (\rho^* \sigma)) & \text{on } U \times N \\ -d(\pi^* \rho_U((\pi^* \omega) \wedge (\rho^* \sigma)) & \text{on } V \times N \end{cases}.$$

Examining the first term, we have

$$\begin{aligned} d(\pi^* \rho_V((\pi^* \omega) \wedge (\rho^* \sigma)) &= d(\pi^*(\rho_V \omega) \wedge (\rho^* \sigma)) \\ &= d(\pi^*(\rho_V \omega)) \wedge (\rho^* \sigma) \pm \pi^*(\rho_V \omega) \wedge \overbrace{d(\rho^* \sigma)}^{=0} \\ &= \pi^*(d(\rho_V \omega)) \wedge (\rho^* \sigma) \end{aligned}$$

as desired.

Definition: Cup Product

For $H^*_{\text{dR}}(M) = \bigoplus_{k=0}^n H^k_{\text{dR}}(M)$, define $\smile : H^k_{\text{dR}}(M) \times H^\ell_{\text{dR}}(M) \to H^{k+\ell}_{\text{dR}}(M)$ by $([\omega], [\eta]) \mapsto [\omega \wedge \eta]$.
Recall that $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$. If ω is closed, then $\eta = d\sigma$ is exact and

$$\omega \wedge \eta = \omega \wedge d\sigma = \pm d(\omega \wedge \sigma) \pm \overbrace{d\omega}^{=0} \wedge \sigma.$$

That is to say that $\omega \wedge \eta$ is exact. This shows that $[\omega \wedge \eta] = [\omega \wedge (\eta + d\sigma)]$. Similarly $[(\omega + d\sigma) \wedge \eta] = [\omega \wedge \eta]$. Hence this product is well-defined.

de Rahm Cohomology Rings

It follows from the definition of the cup product that that $(H^*_{\text{dR}}, +, \smile)$ is a ring where the multiplicative identity is $[1] \in H^0_{\text{dR}}(M)$ where 1 is the constant function on M.

Example

Recall that for S^1, $H^0_{\text{dR}}(S^1) = \mathbb{R}$ and $H^1_{\text{dR}}(S^1) = \mathbb{R} = [\omega]$. Consider $\omega = x\,dy - y\,dx$.

Example

For $\mathbb{T}^2 = S^1 \times S^1$ with a parametric equation $F : [0, 2\pi]^2 \to \mathbb{R}^4 = \{(x, y, z, w)\}$ by $F(t, \theta) = (\cos t, \sin t, \cos\theta, \sin\theta)$, we have that $H^1_{\text{dR}}(M)$ is generated by $\omega = x\,dy + y\,dx$ and $\eta = z\,dw - w\,dz$. Compute

$$\omega \wedge \eta = xz\,dy \wedge dw + yw\,dx \wedge dz - xw\,dy \wedge dz - yz\,dx \wedge dw$$

and

$$F^*(\omega \wedge \eta) = \cos^2 t \cos^2\theta\,dt \wedge d\theta + \sin^2 t \sin^2\theta\,dt \wedge d\theta + \sin^2 t \cos^2\theta\,dt \wedge d\theta + \cos^2 t \sin^2\theta\,dt \wedge d\theta$$

to see that it must be the case that

$$\int_{\mathbb{T}^2} \omega \wedge \eta = \int_{[0,2\pi]^2} F^*(\omega \wedge \eta) > 0$$

and conclude that $[\omega] \smile [\eta] = [\omega \wedge \eta] \neq 0 \in H^2_{\text{dR}}(M)$.

General Cup Product

We can define the cup product on $H^*(X; R)$ for any ring R (but usually $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_m$).
We write $C^k(X) = C^k(X; R) = \operatorname{Hom}(C_k(X), R)$ and let $\varphi \in C^k(X)$, $\psi \in C^\ell(X)$. Then define $\varphi \smile \psi \in C^{k+\ell}(X)$ (by $\sigma : \Delta^{k+\ell} \to X$). So

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \psi(\sigma|_{[v_k, \ldots, v_{k+\ell}]})$$

(where $\cdot$ is the product in R).

Fact

If $\delta : C^k(X) \to C^{k+1}(X)$ is the co-boundary map, then

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi).$$

It follows that $H^k(X) \times H^\ell(x) \to H^{k+\ell}(X)$ by $([\varphi], [\psi]) \mapsto [\varphi \smile \psi]$ is well-defined and $(H^*(X; R), +, \smile)$ is a ring.

Fact

Recall that $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$, so

$$H^i(\mathbb{CP}^n;\mathbb{Z}) \cong H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

The ring structure has a generator $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ since $\alpha \smile \alpha \neq 0 \in H^4(\mathbb{CP}^n;\mathbb{Z})$ and $\alpha^n \neq 0 \in H^{2n}(\mathbb{CP}^n;\mathbb{Z})$. Therefore $H^{2n}(\mathbb{CP}^n) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$.

Example

$S^4 \vee S^2$ has a cell-complex $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$, so

$$H_i(\mathbb{CP}^2) \cong H_i(S^4 \vee S^2) = \begin{cases} \mathbb{Z} & i = 0,2,4 \\ 0 & \text{otherwise} \end{cases}.$$

This fails to differentiate the spaces. However, $H^i(S^4 \vee S^2)$ is generated by $\alpha \in H^2$ and $\beta \in H^4$, but $\alpha \smile \alpha = 0 \neq \beta$. Recall that $\mathbb{CP}^2$ is the gluing of e^4 to $\mathbb{CP}^1 = S^2$ by $\varphi : \partial e^4 = S^3 \to S^2$ where φ is the quotient map of $S^3 \to S^2$ by the circle action $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$. Contrarily, $S^4 \vee S^2$ is the gluing of e^4 to S^2 by way of $\psi = \partial e^4 = S^3 \to \text{pt} \in S^2$. Hence φ is a nontrivial element in $[S^3, S^2] = \pi_3(S^2)$. So we conclude that $\mathbb{CP}^2 \not\simeq S^4 \vee S^2$.

Examples

Simplex

Take the normal construction of the 1-simplex with $\partial\sigma = a + b - c$ and $C_1(X) = \langle a, b, c\rangle$. We consider the dual $\alpha, \beta, \gamma \in C^1(X)$ and

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma|_{[v_0,v_1]}) \cdot \beta(\sigma|_{[v_1,v_2]}) = \alpha(a) \cdot \beta(b) = 1$$

Similarly, $(\alpha \cup \gamma)(\sigma) = \alpha(a) \cdot \gamma(b) = \alpha(a) \cdot 0 = 0$.

Torus

With the usual simplicial construction of $\mathbb{T}^2$, we have a chain complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{d} & \mathbb{Z}^3 & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\ & & \langle U, L\rangle & & \langle a, b, c\rangle & & \langle v\rangle & & \end{array}$$

We get a co-chain complex

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbb{Z}^2 & \xleftarrow[\delta]{} & \mathbb{Z}^3 & \xleftarrow[0]{} & \mathbb{Z} & \longleftarrow & 0 \\ & & \langle \mu, \lambda\rangle & & \langle \alpha, \beta, \gamma\rangle & & \langle \omega\rangle & & \end{array}$$

Then we compute $(\delta\alpha)(U) = \alpha(\partial U) = \alpha(a+b-c) = 1$ and $(\delta\alpha)(L) = \alpha(\partial L) = \alpha(a+b-c) = 1$ so $\delta\alpha = \mu+\lambda$. Similarly, $\delta\beta = \mu+\lambda$ and $\delta\gamma = -\mu-\lambda$, so $\operatorname{im}\delta = \langle \mu+\lambda\rangle$ and $\ker\delta = \langle \alpha+\gamma, \beta+\gamma\rangle$. Therefore $H^1 = \ker\delta = \langle \alpha+\gamma, \beta+\gamma\rangle \cong \mathbb{Z}^2$

and $H^2\langle\mu,\lambda\rangle/\langle\mu+\lambda\rangle \cong \mathbb{Z}$. Examining U as a simplex, we can compute

$$(\alpha \smile \beta)(U) = \alpha(U|_{[v_0,v_1]}) \cdot \beta(U|_{[v_1,v_2]}) = \alpha(b) \cdot \beta(a) = 0$$

so $((\alpha+\gamma) \smile (\beta+\gamma))(U) = 0$. Repeating the process on L,

$$(\alpha \smile \beta)(L) = \alpha(L|_{[v_0,v_1]}) \cdot \beta(L|_{[v_1,v_2]}) = \alpha(a) \cdot \beta(b) = 1$$

which tells us that $(\alpha+\gamma) \smile (\beta+\gamma)$ is the dual of L (i.e. $(\alpha+\gamma) \smile (\beta+\gamma) = \lambda$. Hence $[(\alpha+\gamma) \smile (\beta+\gamma)] \neq 0 \in H^2$. Similarly, $(\alpha+\gamma) \smile (\alpha+\gamma) = 0$ and $(\beta+\gamma) \smile (\beta+\gamma) = 0$.