Manifolds I

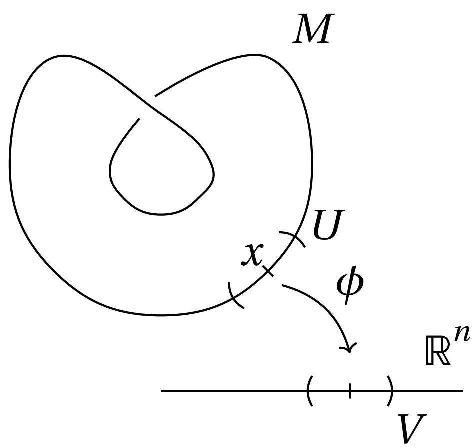
September 26, 2024

Class Organization

1 Takehome Midterm
1 Takehome Final
Homeworks assigned, but not graded.
`https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html`

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M$, $\exists M \supset U \ni x$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ for V open.
To avoid problems (see below), further assume that M is Hausdorff and second countable.

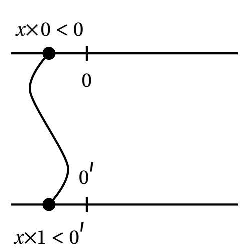


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R}\times 0 \coprod \mathbb{R}\times 1)/\sim$.

- M need not be second countable.

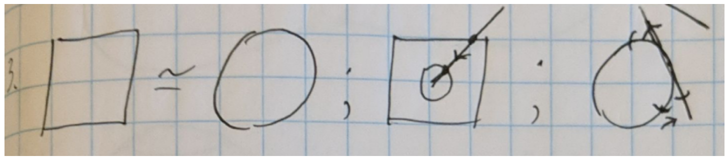
Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

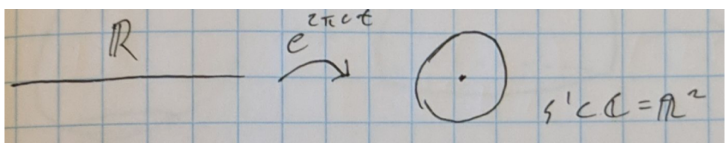
An open subset of a manifold is a manifold.

Example 4

$M,\ N$ manfiolds implies $M \times N$ is a manifold.

Example 5

Take $\mathbb{R}/\mathbb{Z}$ by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



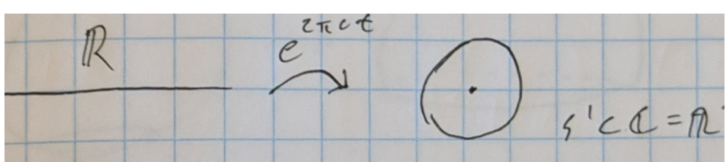
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

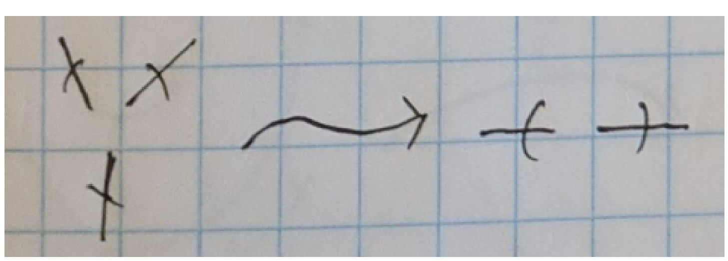
$\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Counterexample 1

$[0, 1]$ is not a manifold.

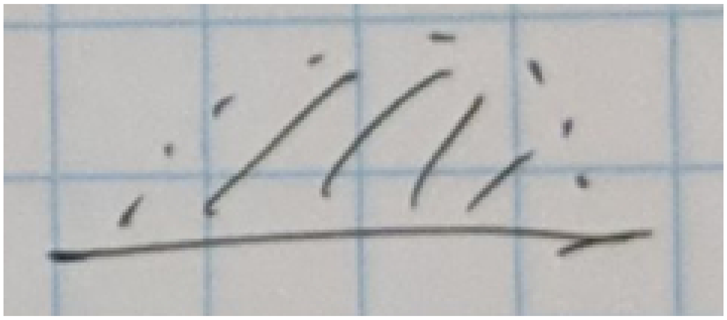


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



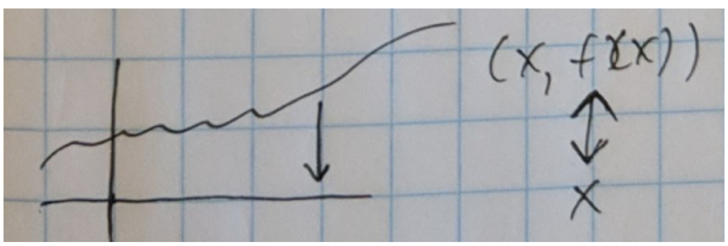
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

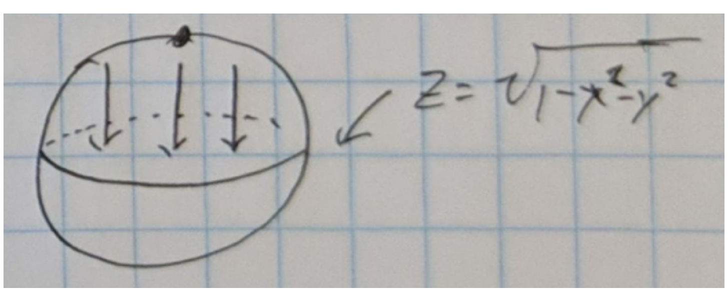


Example 8

Take $f : M \to N$ between manfiolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.



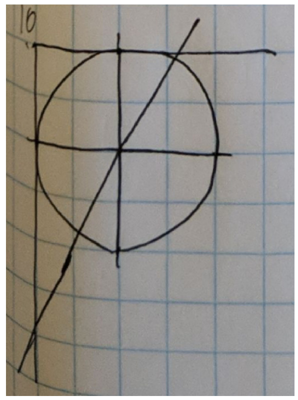
Definition: Real Projective Spaces

Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.

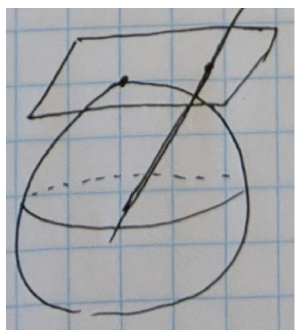
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.
That is, identifiying the antipodal points of the unit sphere.
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

$\mathbb{RP}^n$ is a manifold.

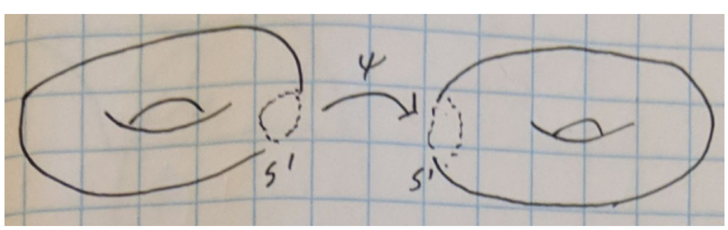


$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \overset{\text{homeo}}{\to} \mathbb{R}$.

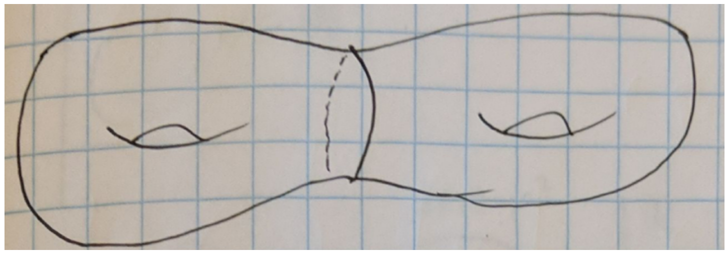


$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \overset{\text{homeo}}{\to} \mathbb{R}^2$
We have that $\mathbb{RP}^1$ is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.
Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \cdots : x_n]$.
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] \, : \, x_k \neq 0\}$, we have that $U_0, \dots, U_n$ covers $\mathbb{RP}^n$.
Then define $U_k \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \to \left(\frac{x_0}{x_k}, \dots, \frac{x_k}{x_k}, \dots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manfiolds

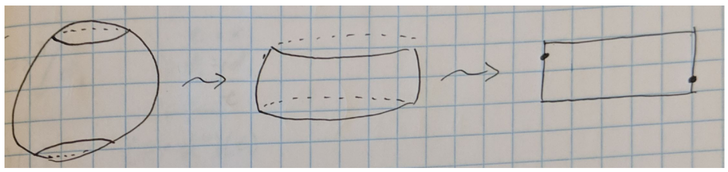


$M \setminus B^n \coprod N \setminus B^n$



$M \# N$.

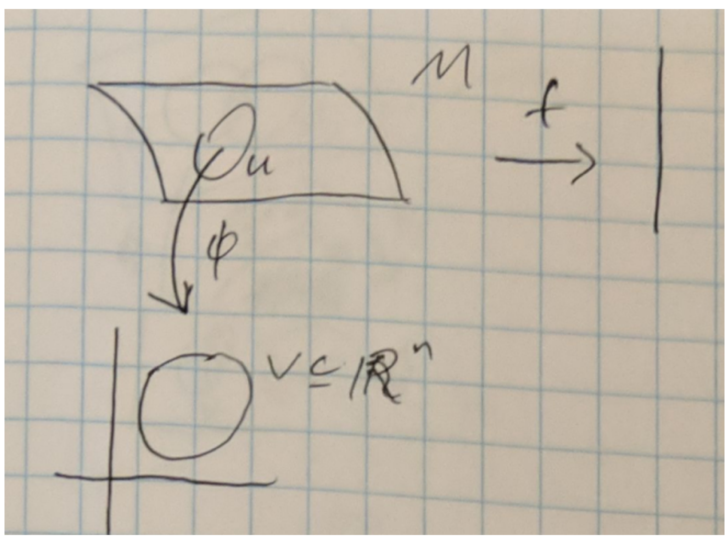
Mobius Band



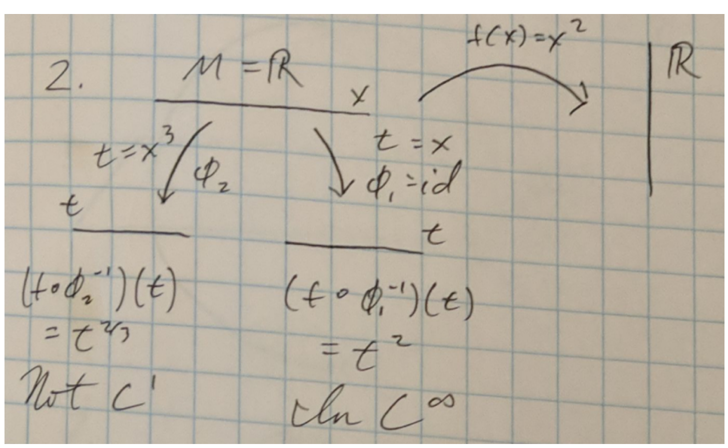
October 1, 2024

A Failed Definition

$f \in C^{r \geq 1}$; $f \circ \phi^{-1} : V \overset{C^r}{\to} \mathbb{R}$.



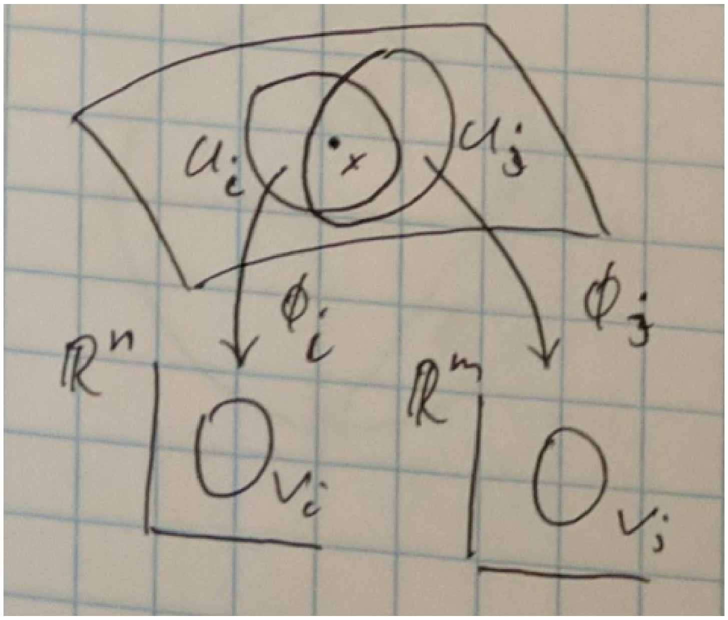
Example



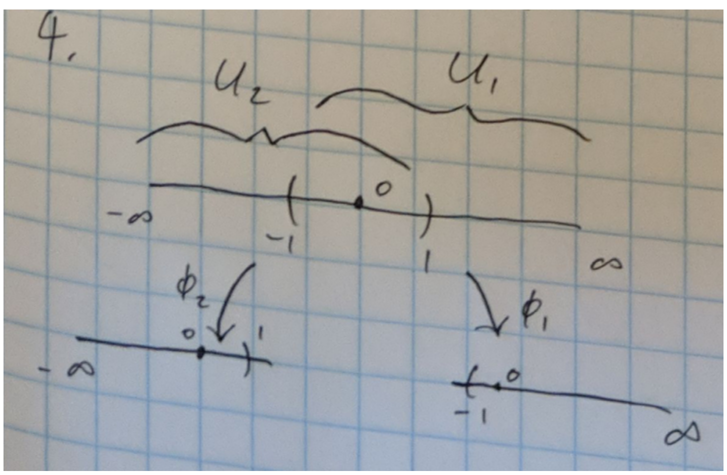
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x, $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

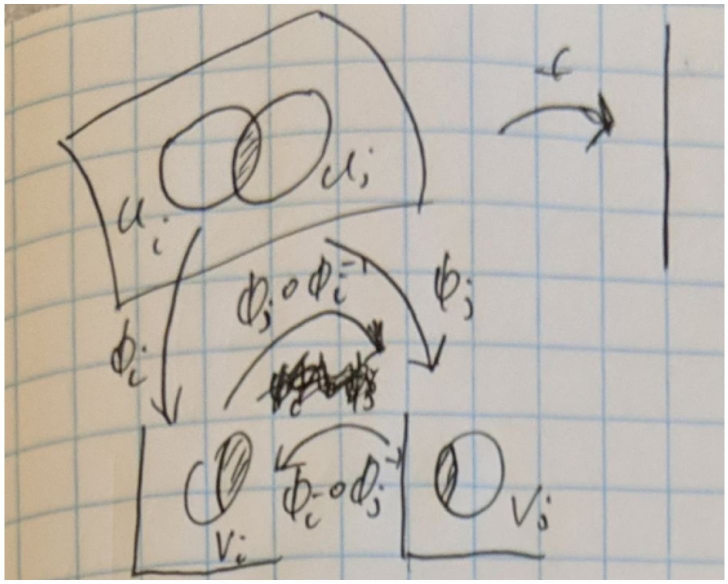


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j.



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$.
Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2.
Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider $\mathbb{R}$.
Atlas 1: $U = \mathbb{R}$ and $\phi = \mathrm{id}$.
Atlas 2: $U_1 = (1, \infty)$, $\phi_1 = (x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homemorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

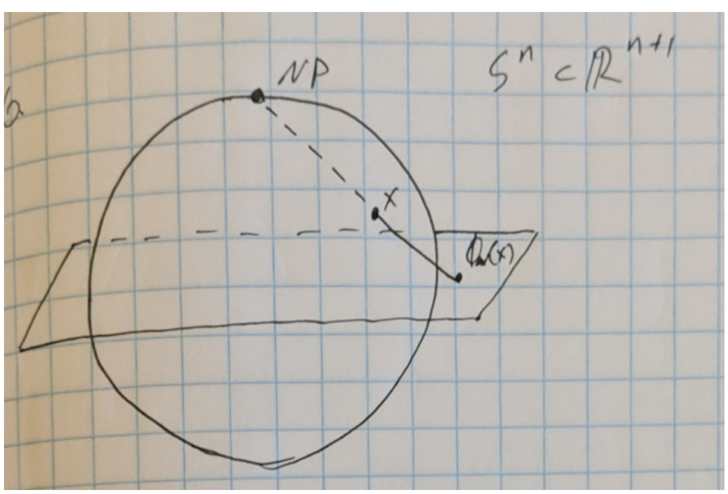
- Zero dimensional manfiolds (i.e. a point).
- $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.
- If M, N are smooth manfiolds, then $M \times N$ is a a smooth manfiold.

  That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j), we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.
- Take $F : M \overset{\text{homeo}}{\to} N$ with N a smooth manfiold. Then M is smooth.

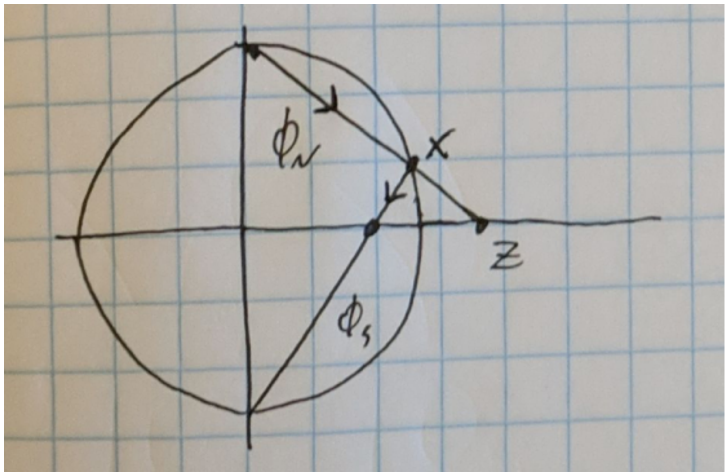
  Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.
- An open subset of a smooth M is a smooth manifold.
- $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \overset{\phi_N}{\to} \mathbb{R}^n$$
$$U_S = S^n \setminus SP \underset{\phi_S}{\to} \mathbb{R}^n$$



$\phi_S \phi_N^{-1}(z) = \frac{z}{|z|^2}$.

- A different construction for S^n.

  Take hemispheres $U \xrightarrow{\text{orthogonal projection}} B^n$.

Projective Space

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim$ where $x \sim \lambda x$ for $\lambda \neq 0$.
$[x] = [x_0 : x_2 : \cdots : x_n] = [\lambda x_0 : \lambda x_2 : \cdots : \lambda x_n]$.
Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \to \mathbb{R}^n$ given by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. Then for $j < i$ take

$$\phi_j \phi_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_0}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_1}, \frac{1}{y_i}, \frac{y_i}{y_i}, \ldots, \frac{y_n}{y_1}\right)$$

Definition: Diffeomorphism

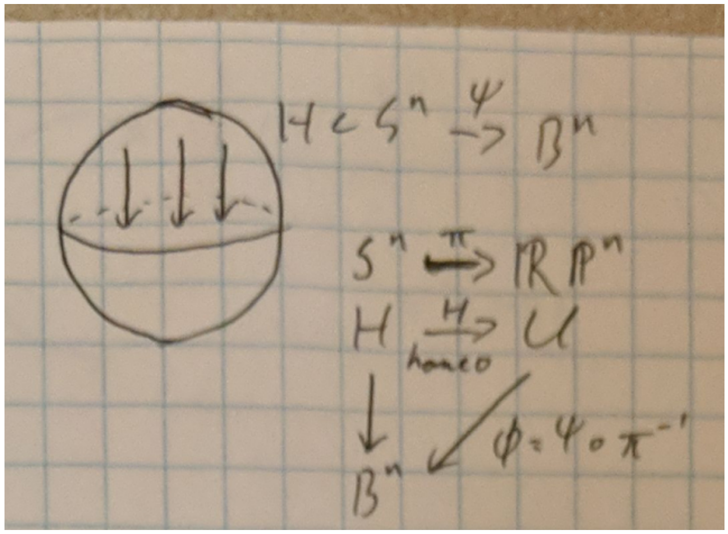
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\text{max}} & & A \supset A_{\text{max}} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1} A_{\text{max}} = B_{\text{max}}$ ($F^{-1} A \sim B$).

October 3, 2024

Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0 / \sim & x \mapsto \lambda x \\ S^n / x \sim -x \end{cases}$$



Note

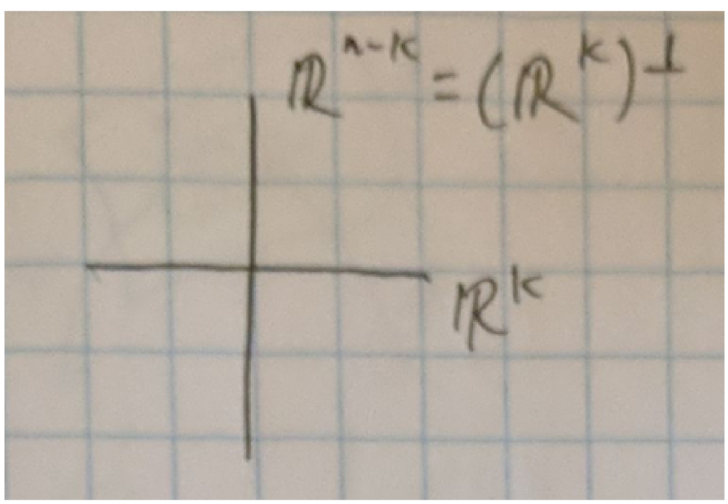
Given a manifold M and A a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given $M \underset{\text{homeo}}{\overset{F}{\to}} M$, $F^{-1}A \nsim A$.

Confer With Groups

$G \xrightarrow{F} G$, $a * b = F^{-1}(F(a)F(b))$.

Definition: Grassmannians

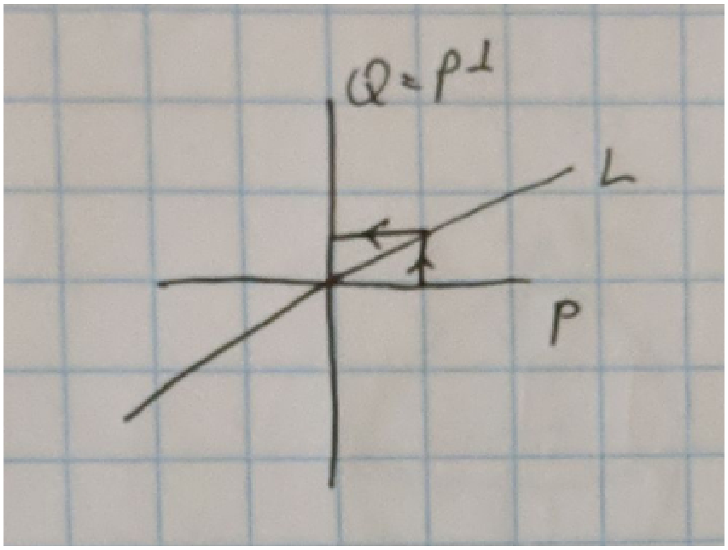
Write $G_k(n)$, the collection of all k-dimensional subspace L in $\mathbb{R}^n$.



Observe that if $O(i)$ is the collection of orthogonal transformations in dimension i,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with $X \sim Y$ when $Y = XA = X(O(K) \times O(n-k))$.

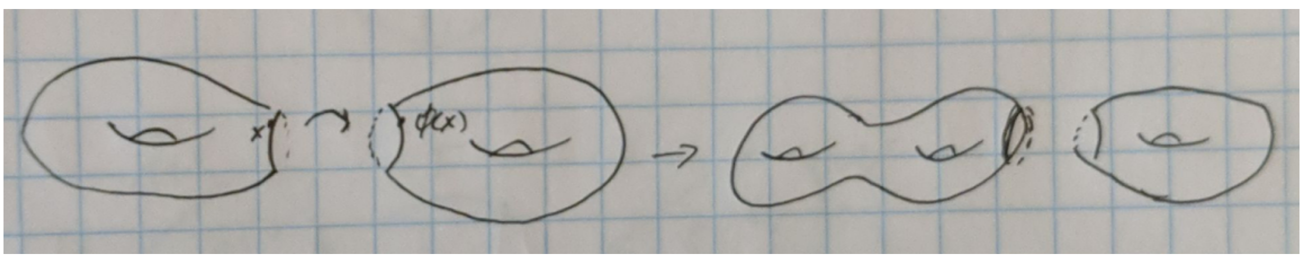


Where $\dim(L) = k$, $U_p = \{L \,:\, L \cap Q = \{0\}\}$, $L = \mathrm{graph}(A : P \to Q)$, and we have a homemorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \to Q\}}_{\mathbb{R}^{k\times(n-k)}}.$$

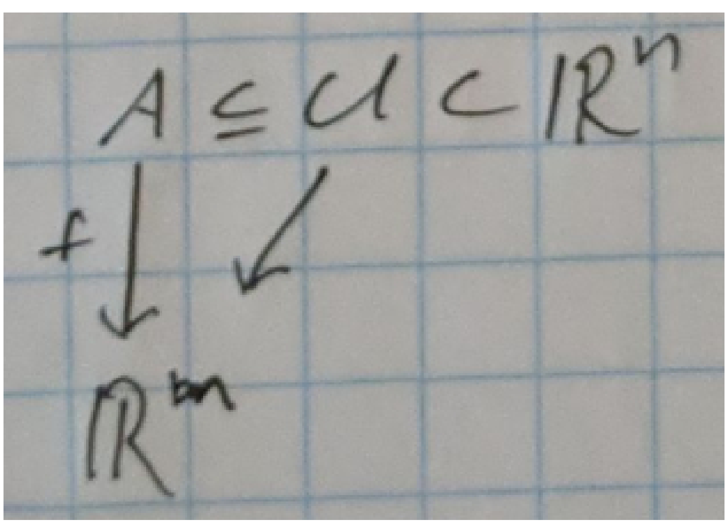
Surfaces

We have explored S^2, $\mathbb{RP}^2$, $\mathbb{T}^2 = S^1 \times S^1$. We have also connected sums.



Terminological Remark

Let $\mathbb{R}^N \supset A \xrightarrow{f} \mathbb{R}^m$.
Then f is smooth if it extends to a smooth map



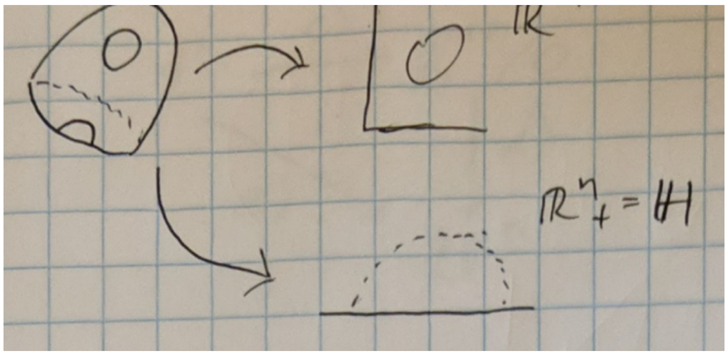
Exercise

Let $A = [0, \infty) \subset \mathbb{R}$. $f : A \to \mathbb{R}$ is smooth if and only if it is infinitely differentiable.
Construct $(-\varepsilon, \infty)$.

Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas A with charts of two types

$$\phi : U \to B^n \quad \text{(open ball)}$$
$$\phi : U \to B^n \cap H$$



As before, $\phi_i \circ \phi_j^{-1}$ must be smooth.

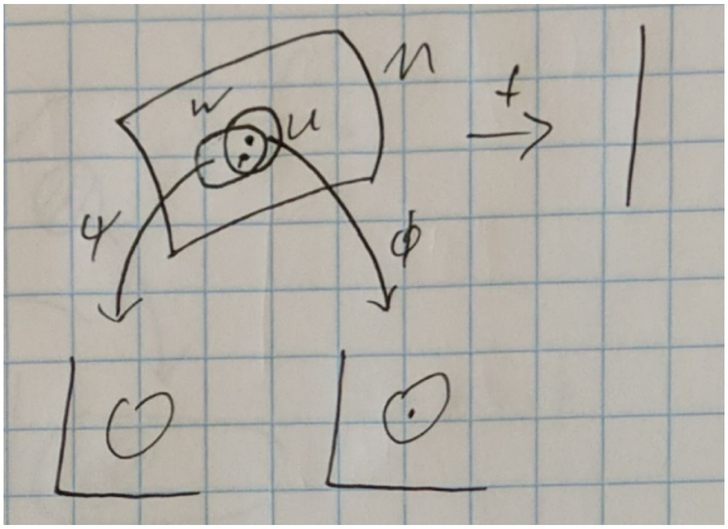
Examples

- $M \backslash$open ball.
- The upper half space.

Definition-Lemma: Smooth Function

A function $f : M \to \mathbb{R}$ is smooth at $p \in M$ if either of the following equivalent conditions is satisfied

1. $\exists$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.
2. $\forall$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.



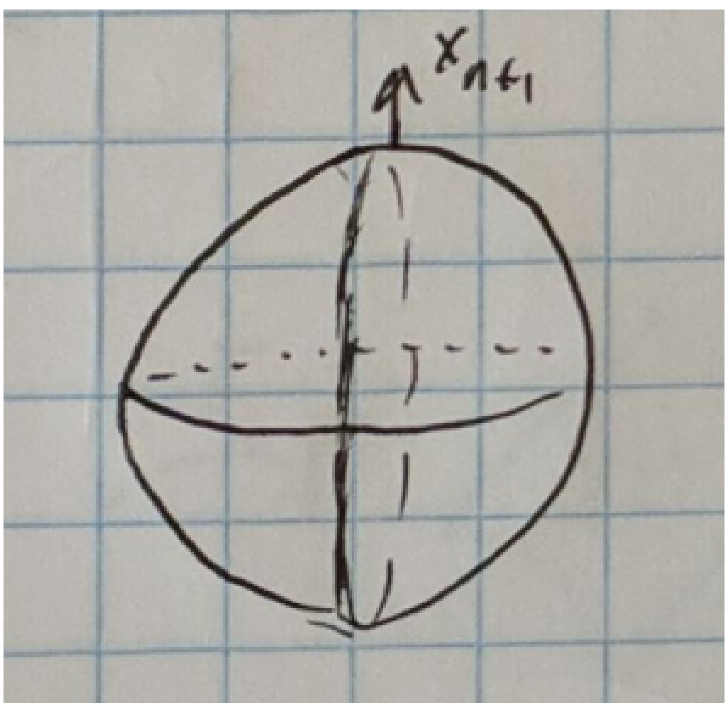
Where $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$.
If the above hold for each $p \in M$, then f is smooth.

Remark

f smooth implies f is C^0

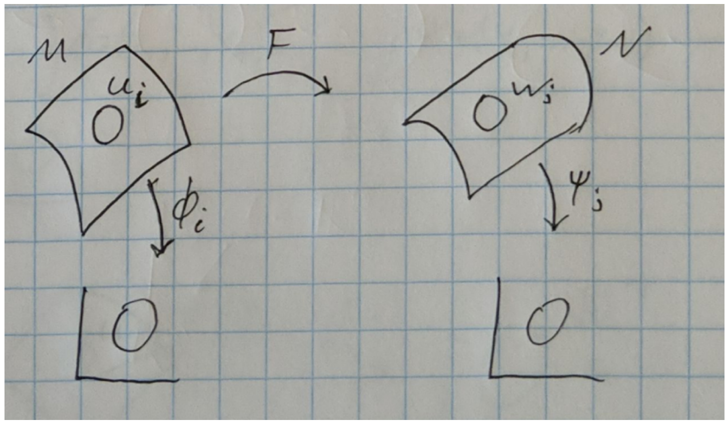
Exercise / Sketch

The height function on S^n is smooth.



$\phi : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$.
$f \circ \phi^{-1} = \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}$.
Note that handling the equator requires examining the Eastern and Western hemisphere.
The stereographic projection leads to a simpler proof.

Definition: Smooth Function Between Manifolds



$F : M \to N$ is smooth if F is C^0 and one of the following equivalent conditions is satisfied

1. $\exists$ an atlas $A \subset A_{\max}$ on M and an atlas $B \subset B_{\max}$ on N such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth on $F^{-1}(W_j) \cap U_i$.

2. The same as a., but for $A_{\max}$ and $B_{\max}$.

Consider as an example $S^n \to \mathbb{RP}^n$.

Properties of Smooth Maps

$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0$.
The sum and product of smooth functions is smooth.

Exercise

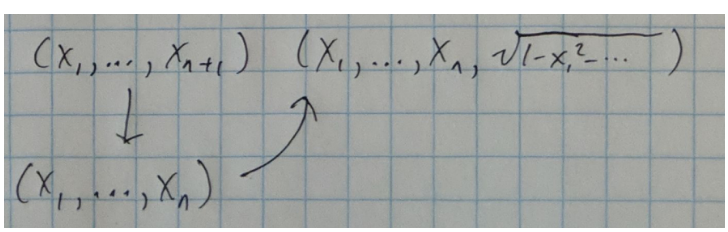
The composition of smooth maps is smooth.

Examples

1. $M \times N \overset{pr}{\to} M$ is smooth.

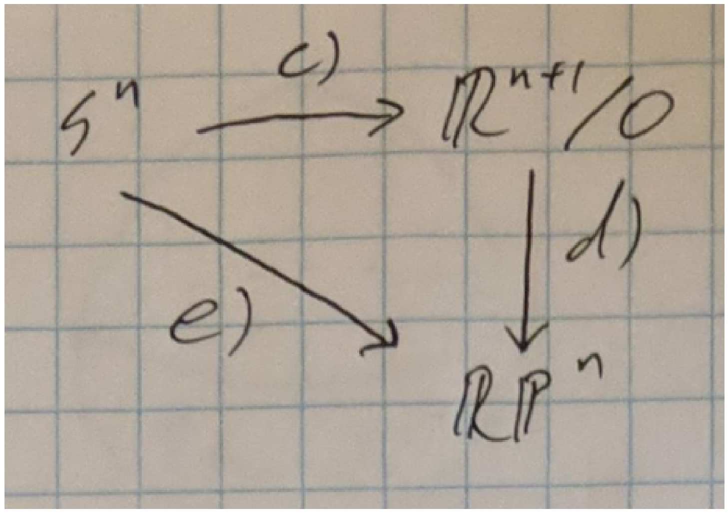
2. $\underbrace{M \overset{(F_1,F_2)}{\to} N_1 \times N_2}_{\text{smooth}}$ if and only if F_1 and F_2 are smooth.

3. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.



1. $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is smooth with $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}\right)$.

2. $S^n \to \mathbb{RP}^n$.



Definition: Diffeomorphism

$F: \underset{A_{\max}}{M} \xrightarrow{\text{diffeo}} \underset{B_{\max}}{N}$ if

- F is smooth.
- F is invertible.
- F^{-1} is smooth.

Previous Definition

$F^{-1}(B_{\max}) = A_{\max}$.

Exercise

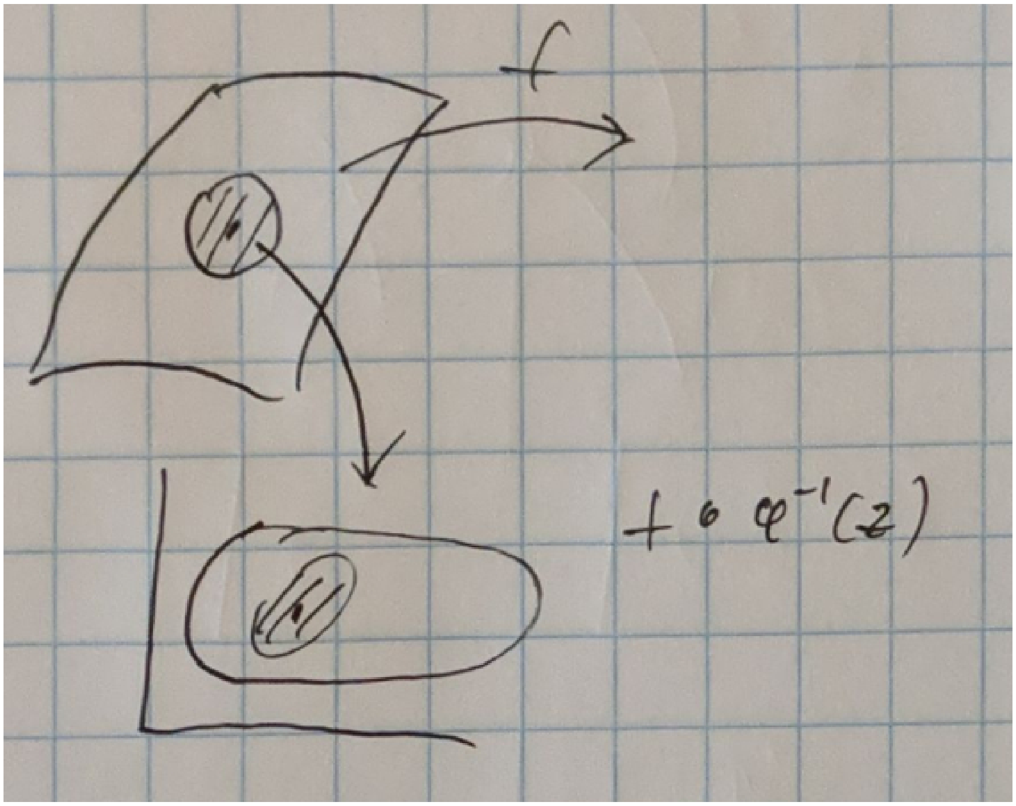
Prove that the definitions are equivalent.

Examples

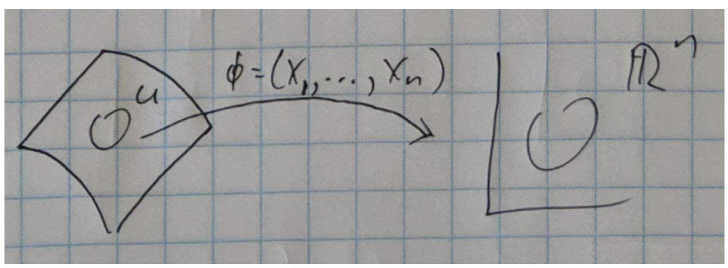
1. $\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$.
2. $x \mapsto x^3$, $\mathbb{R} \to \mathbb{R}$ is not a diffeomorphism.
3. $G_k(n) \leftrightarrow G_{n-k}(n)$ with $P \leftrightarrow P^{\perp}$.

Example 4

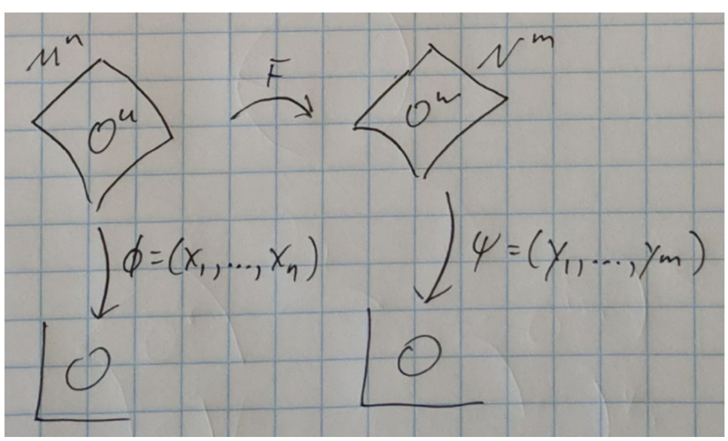
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



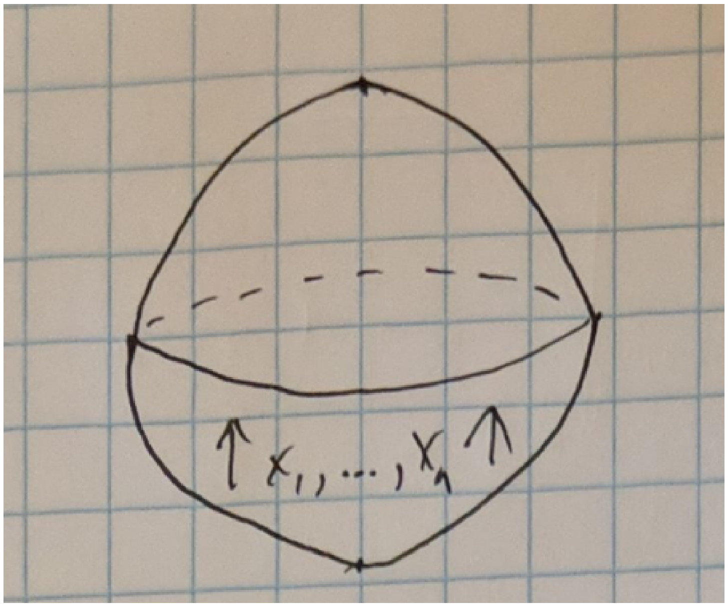
Example 5



Where $\phi = (x_1, \dots, x_n)$ and each x_i is a real-valued function.



$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n))$.
Then $S^n \hookrightarrow \mathbb{R}^{n+1}$



where $y_1 = x_1, \dots, y_n = x_n$, and $y_{n+1} = -\sqrt{1 - x_1^2 - \cdots - x_n^2}$.

Example 6

$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n$.
Need to check that $\psi_j \circ F$ is smooth.
$[t_0 : \cdots : t_n] \xrightarrow{\psi_0} \left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right)$ with $U_0 : t_0 \neq 0$

October 8, 2024

Questions

Question 1

Given M smooth and $x \neq y \in M$, does there exist $f \in C^r(M)$ such that $f(x) = 0$ and $f(y) = 1$.

Question 2

Given $K \subset U \subset M$ with K compact and U open and $g : K \xrightarrow{C^r} \mathbb{R}$, does there exist a C^r extension f of g on M such that $\operatorname{supp} f \subset U$.

Definition: Partitions of Unity

Let W_i be a locally finite open cover.
A partition of unity subordinated to W_i, is a collection of functions $f_i : M \xrightarrow{C^r} \mathbb{R}$ satisfying

- $0 \leq f_i \leq 1$
- $\operatorname{supp} f_i \subset W_i$
- $\sum f_i \equiv 1$

Definition: Refinement

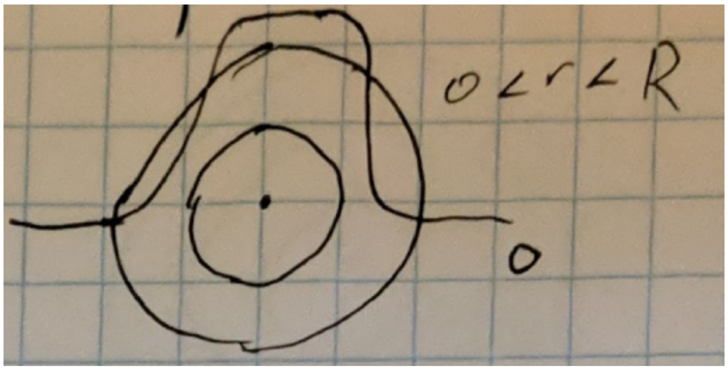
Given covers U_j and W_i, U_j is a refinement if for each j we may find i such that $U_j \subset W_i$.

Theorem

There exists a partition of unity subordinated to W_i.

Lemma 1

Take $B(r) \subset B(R) \subset \mathbb{R}^n$.



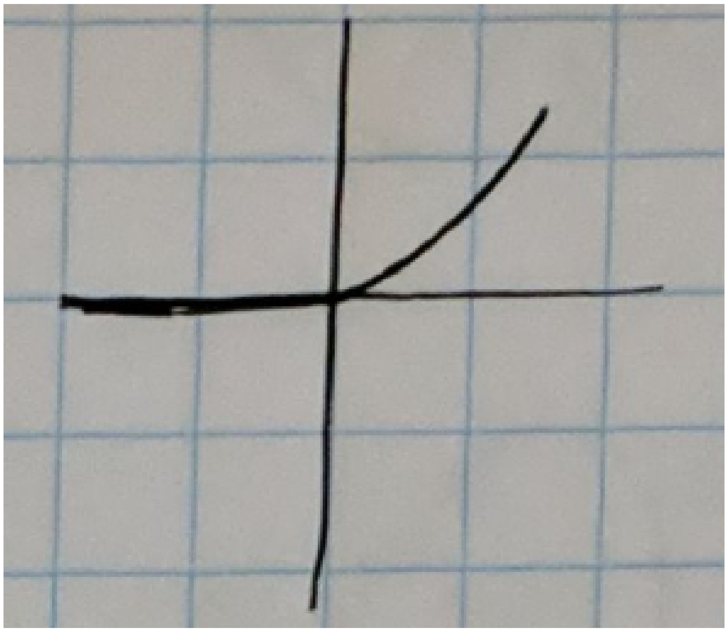
Then there exists $g \in C^\infty$ such that

- $0 \le g \le 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\operatorname{supp} g \subset B(R)$

Proof

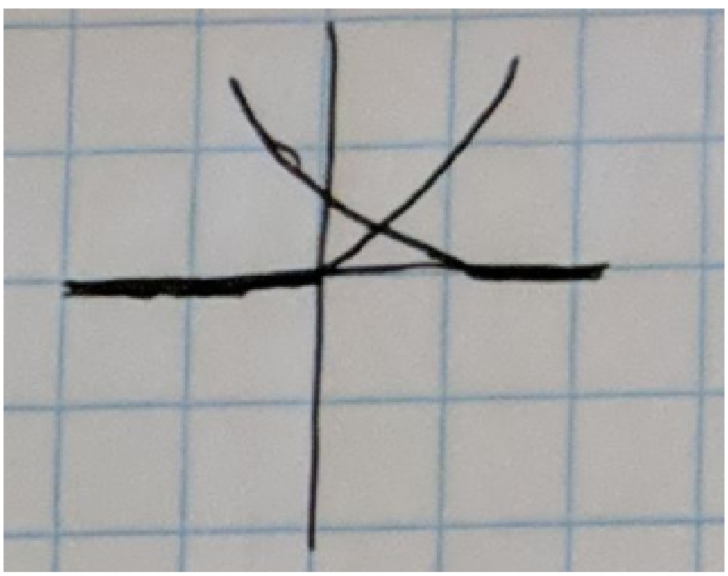
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$



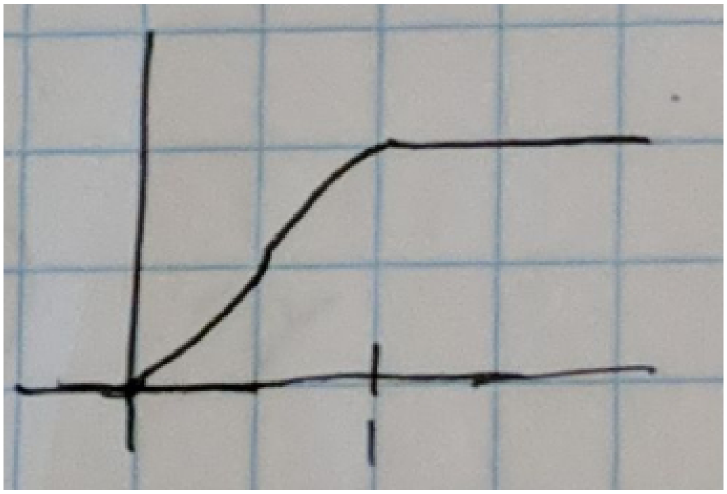
- Exercise: show that $h_0 \in C^\infty(\mathbb{R})$ (Hint: show that derivatives agree at zero from the right.)

Then take $h_1(x) = h_0(x) \cdot h_0(1-x)$.

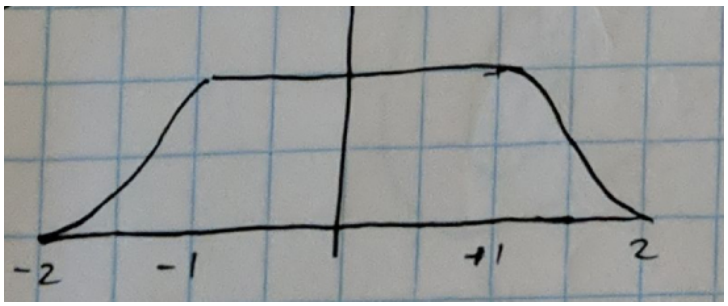


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t)\,dt} \int_{-\infty}^{x} h_1(t)\,dt$$



Finally, we define $h(x) = h_2(x+2) \cdot h_2(2-x)$.



Then $g(x) = h(||x||)$ satisfies our requirements.

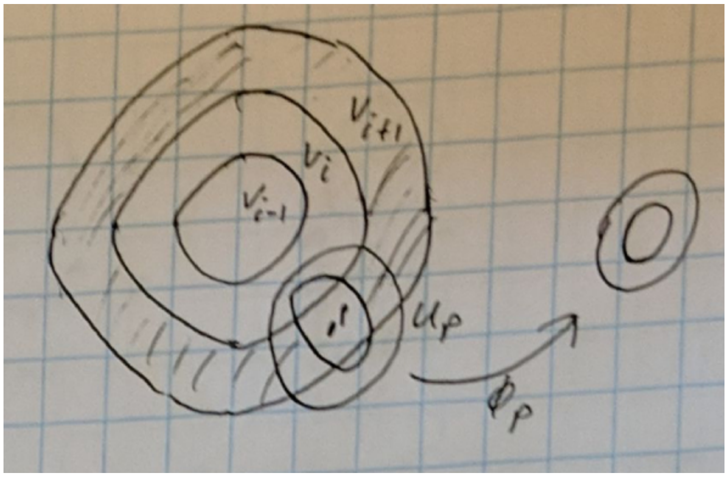
Lemma 2

There exists a refinement (U_j, ϕ_j) by coordinate charts such that

1. U_j is locally finite.
2. $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$.
3. $\phi_j^{-1}(B(1))$ is also a cover.

Proof

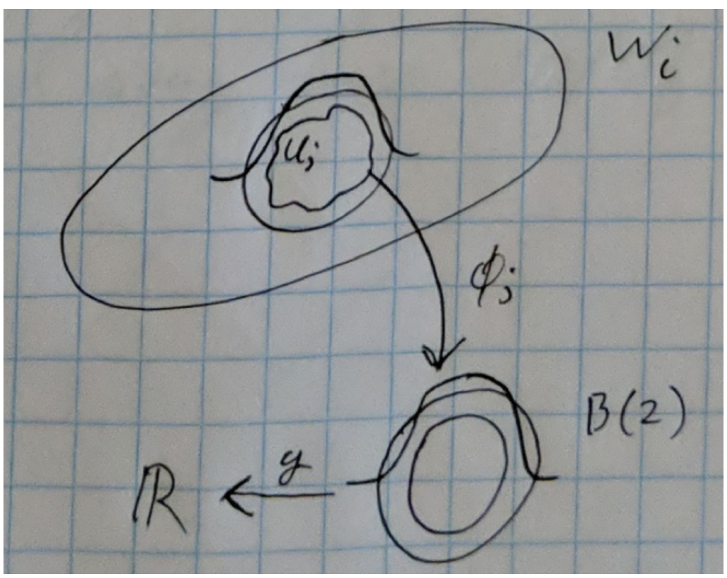
There exists a compact exhaustion of M, $C_1 \subset C_2 \subset C_3 \subset \cdots$ where $\bigcup C_i = M$.
There exists also an open exhaustion by precompact open sets $\varnothing = V_0 \subset V_1 \subset V_2 \subset \cdots$ where $\overline{V_i} \subset V_{i+1}$.
Then define $V_i := \mathrm{nbhd}(C_i \cup \overline{V}_{i-1})$.



Take $A_i = \overline{V}_{i+1} \setminus V_i$ compact, $p \in A_i$. Then we have a map $U_p \xrightarrow{\phi_p} B(2)$.

- $U_p \subset W_i$ for some i. (Refinement)
- $U_p \cap V_{i-1} = \varnothing$. (Locally Finite)
- There exists a finite subcover such that $\phi_p^{-1}(B(1))$ is also an open cover.

Proof of Theorem



Set $g_j = g \circ \phi_j \in C^r$, extended to 0 outside of U_j.

- $\operatorname{supp} g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\operatorname{supp} g_j$ is locally finite.

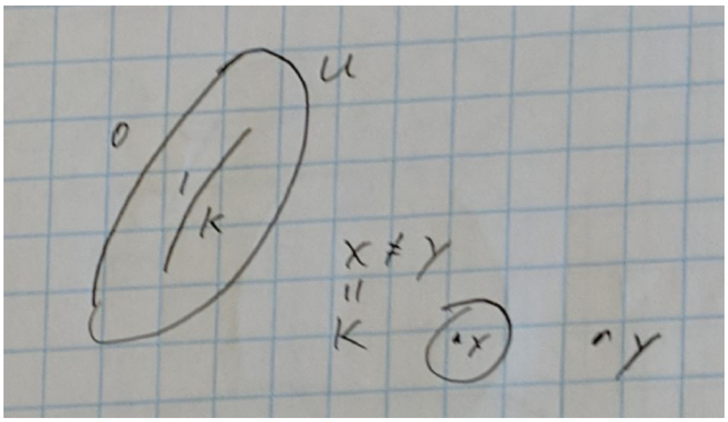
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

Corollary 1

For $K \subset U$, K compact and U open, there exists $h \in C^r(M)$ such that

- $h|_K \equiv 1$
- $\operatorname{supp} h \subset U$



Proof

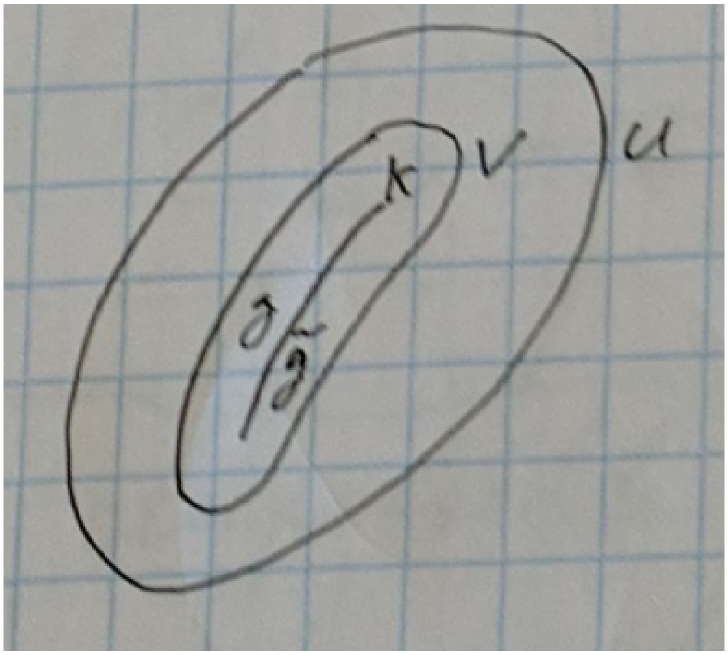
Take $U_0 = U$ and $U_1 = M \setminus K$.
Then there exists a partition of unity f_0 and f_1 where $f_0 + f_1 = 1$. Therefore f_0 has support in U and f_1 has support in $M \setminus K$ this occurs if and only if $f_1|_K = 0$.

Corollary 2

For $K \subset U$, K compact and U open, and $g : K \overset{C^r}{\to} \mathbb{R}$, there exists an extension $f : M \to \mathbb{R}$ of g: $f|_K = g$ such that $\operatorname{supp} f \subset U$.

Proof

For $g \in C^r(K)$, there exists a neighborhood V of K and a function $\tilde{g} : V \to \mathbb{R}$ such that $\tilde{g}|_K = g$.



By corollary 1, there exists $h \in C^r(M)$ such that

- $\operatorname{supp} h \subset V$
- $h|_K = 1$

Therefore $f = h \cdot \tilde{g}$.

Corollary 3

There exists $f : M \overset{C^r}{\to} \mathbb{R}$ bounded from below and proper.

Definition: Proper Function

f is proper if and only if $f^{-1}(K)$ is compact for K compact.

Proof

See Textbook.

Consequence

Take $\{x : f(x) \leq C_i\} =: E_i$ as $i \to \infty$. Then we get a compact exhaustion $E_1 \subset E_2 \subset E_3 \subset \cdots$.
e.g. $f(x) = ||x||^2$.

October 10, 2024

Algebra = Analysis (Geometry)

Take M to be either a compact metrizable space ($A = C^0(M)$) or a copact manifold ($A = C^\infty(M)$).

$$M \leftrightarrow A$$

Let $I \subset A$ be an ideal and take $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$.
Take also $Y \subset M$ closed and consider $I_Y = \{f : f|_Y = 0\}$ which is also an ideal.

$$Y \to I_y$$
$$V(I) \leftarrow I$$

Denote $I_x = \{f : f(x) = 0\}$.

Theorem

- I_x is a maximal ideal.
- Every maximal ideal is of this form, and x is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

Proof

Maximal Ideal

Take I_x and $g \notin I_x$.
We want to show that g with I_x generates A.
Then take $f \in A$ defined as $f = h + ag$ for some $h \in I_x$.
Since $g \notin I_x$, $g(x) \neq 0$. Take $a = \frac{f(x)}{g(x)}$ and define $h := f - ag$.
Then $f(x) - \frac{f(x)}{g(x)} g(x) = 0 \in I_x$.

Every Maximal Ideal

Let I be a proper ideal such that $V(I) \neq 0$, $\exists x$ such that $f(x) = 0$, $\forall f \in I$.
Maximal $\implies V(I) = \{x\}$.
By contradiction, assume not: $\forall x \in M$, $\exists f_x \in I$, $f_x(x) \neq 0$.
It follows that $f_x|_{U_x} \neq 0$ where $U_x \ni x$ is a neighborhood from an open cover.
Then, by compactness, we have a finite subcover
$I \ni f_i = f_{x_i} \neq 0$ on U_i.
$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$ on M. But then $1 = g^{-1} g \in I$.

Uniqueness

$I_{x_1} = I_{x_2} \iff x_1 = x_2$
($\Longleftarrow$) is obvious.
($\Longrightarrow$) $x_1 \neq x_2$ implies that there exists $f \in A$ such that $f(x_1) = 1$ and $f(x_2) = 0$.
Then $f \in I_{x_2}$ while $f \notin I_{x_1}$ implies that $I_{x_1} \neq I_{x_2}$.

Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

Algebra Homomorphisms

Take $\phi : A \to \mathbb{R}$. Then $\ker\phi$ is a maximal ideal, and

$$A/I_x \xrightarrow{\delta_x} \mathbb{R}$$
$$x \mapsto \pm f(x)$$

Then

$$M \leftrightarrow \text{Alg. Hom. } A \to \mathbb{R}$$
$$1 \mapsto 1$$
$$A > 0 \mapsto \text{pos. \#}$$

Counterexample

Take instead $M = \mathbb{R}$. Claim: there exist a maximal ideals other than I_x.

Proof

Take J to be the ideal of all compactly supported functions such that $J \subset M$ for some maximal ideal M. However, $\forall x \in \mathbb{R}$ there exists a compactly supported function $f(x) \neq 0$. So $M \neq I_x$.

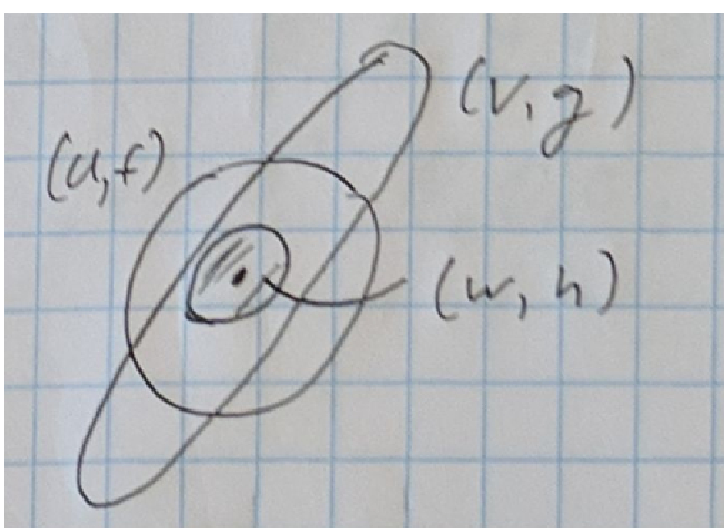
Functorial Considerations

Take $F : M \to N$, M and N compact. Then

$$F : M \to N$$
$$C^{0/\infty}(M) \underset{F^A}{\leftarrow} C^0(N)$$
$$f \circ F \leftarrow\!\shortmid f$$

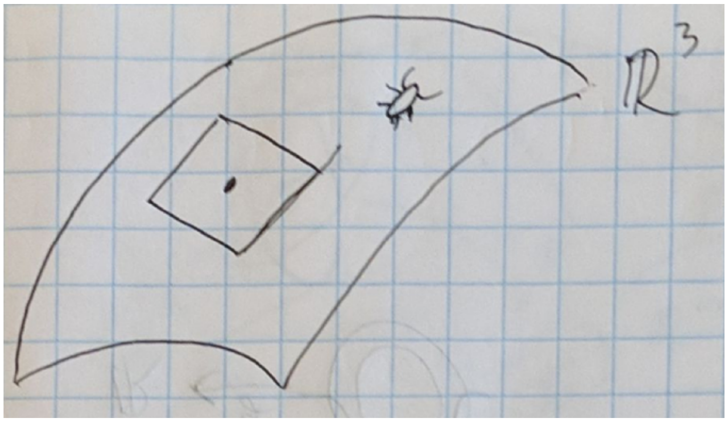
Definition: Germ

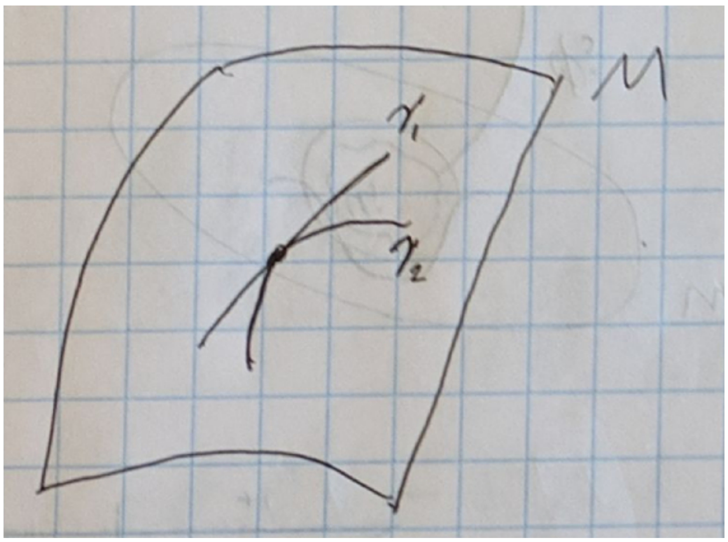
Take $M \ni P$.



Where $(U, f) \sim (V, g)$ if $f \equiv g$ on some neighborhood of p.

Definition: Tangent Spaces



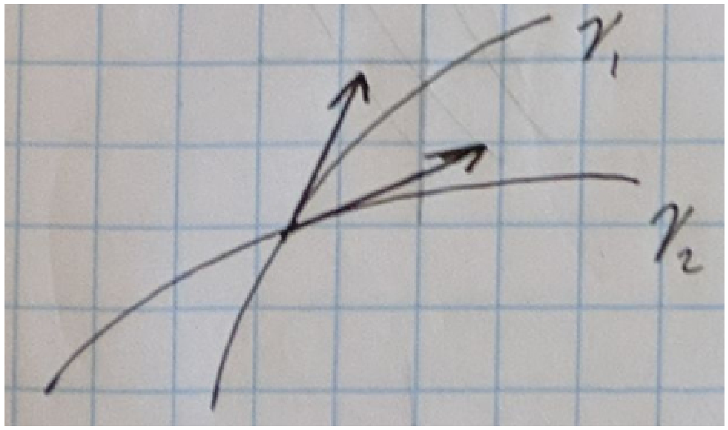


With $\gamma_1 : (-\varepsilon, \varepsilon) \to M$, $\gamma_2(-\varepsilon, \varepsilon) \to M$, and $\gamma_1 \sim \gamma_2 \iff \gamma_1'(0) = \gamma_2'(0)$.
$\gamma(t) = ((x(t), y(t), z(t))$ and $\gamma'(0) = (x'(0), y'(0), z'(0))$.
The tangent space to M at p, written T_pM, is the set of equivalence classes.

Remarks: Tangent Vectors

Take $V \ni p$ a finite-dimensional vector space ($= \mathbb{R}^n$).

$$0 \mapsto p$$
$$\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to V \quad \text{(germs)}$$
$$\gamma_1' \sim \gamma_2' : \gamma_1'(0) = \gamma_2'(0)$$



Write

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \ldots, x_n)$ and $\gamma'(0) = (x_1'(0), \ldots, x_n'(0))$.
A tangent vector to V at p is an equivalence class $T_p V$.
Claim: $T_p V$ is a vector space.

Operations

Take $p = 0$. Write

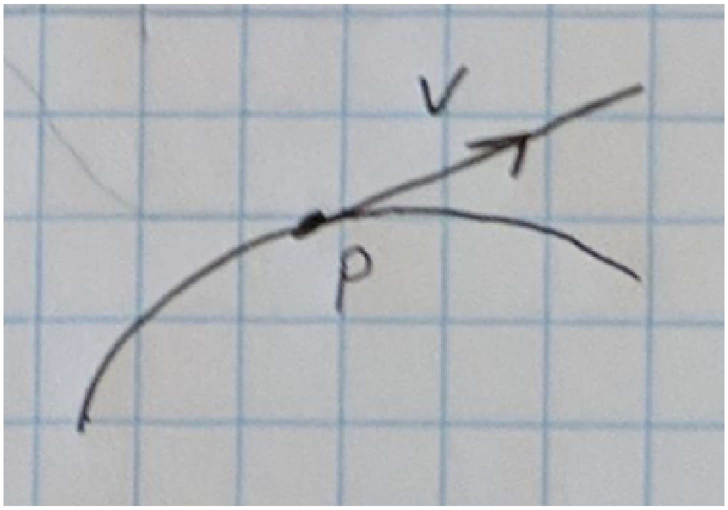
$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2]$$
$$\lambda[\gamma] = [\lambda\gamma]$$

When $p \neq 0$, instead

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2 - p]$$
$$\lambda[\gamma] = [\lambda\gamma + (1 - \lambda)p]$$

Claim: $T_p V$ is canonically isomorphic to V

$$[\gamma] \mapsto \gamma'(0)$$
$$[\gamma = (x_1, \ldots, x_n)] \mapsto (x_1'(0), \ldots, x_n'(0))$$
$$T_p V \to V$$
$$p + tv \leftarrow\!\shortmid v$$



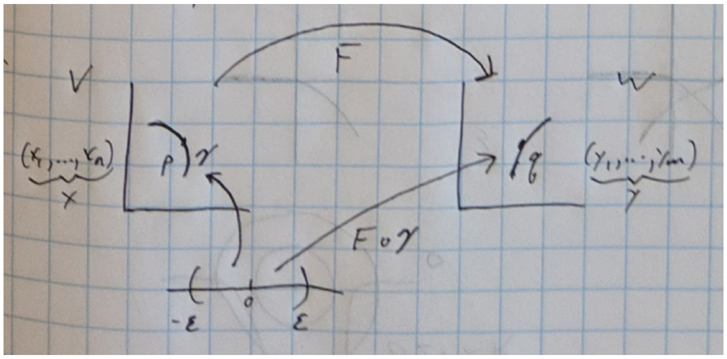
Proposition

Take

$$\underset{\mathbb{R}^n}{V} \xrightarrow{C^r} \underset{\mathbb{R}^m}{W}$$
$$p \mapsto q$$

Then

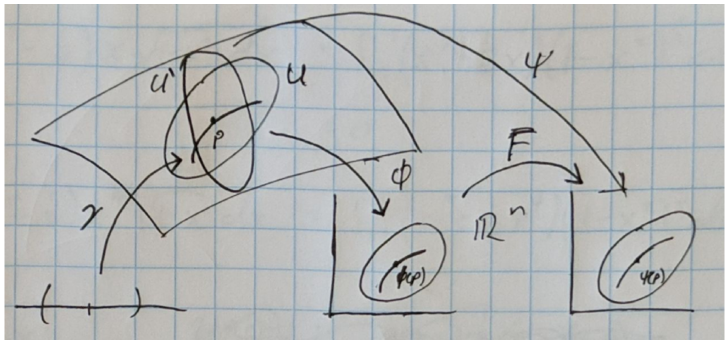
$$V \xrightarrow{\sim} W$$
$$DF_p = F_* : T_pV \to T_qW$$
$$[\gamma] \mapsto [F \circ \gamma]$$



Then F is well-defined and linear.
$F = (F_1, \ldots, F_m)$ and $F \circ \gamma(F_1(\gamma_1, \ldots, \gamma_n), \ldots, F_m(\gamma_1, \ldots, \gamma_n)$.
We have that $[\gamma] = \gamma^{'}(0)$ and $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$. By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma_1^{'}(0) \\ \vdots \\ \gamma_n^{'}(0) \end{pmatrix}$$

Tangent Space



$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$ and $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$, so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take $\{[\gamma]\} = T_pM$. Claim: this is a vector space.

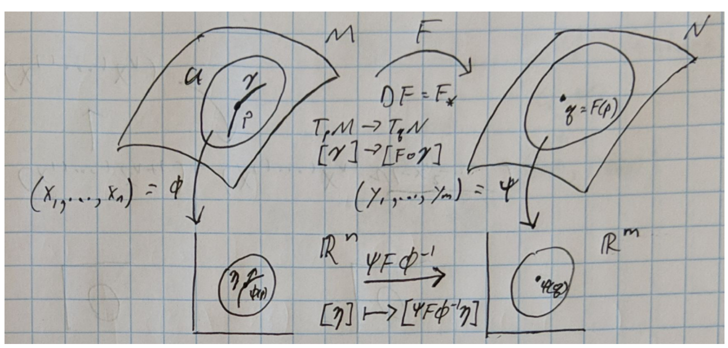
$$\begin{tikzcd} T_pM \arrow[r, "D\phi"] \arrow[dr, "D\psi"'] & T_{\phi(p)}\mathbb{R}^n \arrow[d, "D(\overbrace{\psi\circ\phi^{-1}}^{F})"] \\ & T_{\psi(p)}\mathbb{R}^n \end{tikzcd}$$

$$[\gamma_1] + [\gamma_2] = \left[\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)\right].$$

October 15, 2024

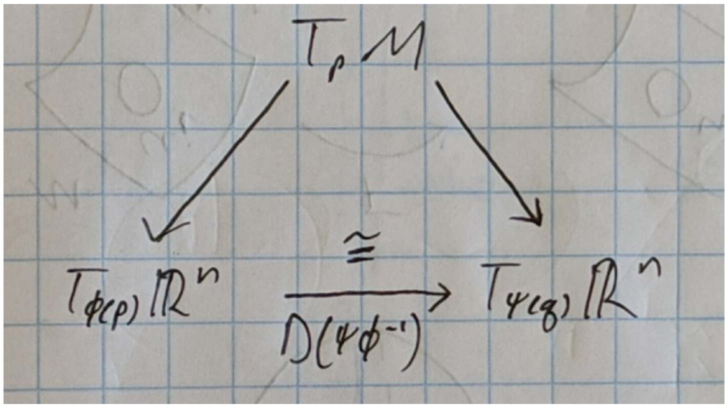
Recall: Tangent Space by Equivalence Classes

$\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, $\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1' \sim \phi \circ \gamma_2' \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$.
$T_pM = \{[\gamma]\} \overset{D\phi = \phi_*}{\to} T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n$.



Then for $(F_1, \ldots, F_m)$, we have $D(\psi F \phi^{-1}) : \underbrace{T_{\phi(p)}\mathbb{R}^n}_{\mathbb{R}^n} \to \underbrace{T_{\psi(q)}\mathbb{R}^m}_{\mathbb{R}^m}$ where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



Chain Rule

We have that $D(FG) = DF \circ DG$ since $(FG) \circ \gamma = F(G \circ \gamma)$.

Example

Take

$$f : \underbrace{M}_{(x_1,\ldots,x_n)} \to \mathbb{R}$$

and $Df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

Definition: Directional Derivatives

Take $v = [\gamma] \in T_pM$ and $f \in C^r(M)$.
The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1})}_{g} \underbrace{(\phi \circ \gamma)}_{\eta}(t)|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi)\eta_i'(0)$$

which is determined by $\left(\eta_1'(0), \ldots, \eta_n'(0)\right) = \eta'(0)$.

Properties

$L_v : C^\infty(M) \to \mathbb{R}$

1. Linear over $\mathbb{R}$

2. $L_v(fg) = (L_v f)\, g(\phi) + f(\phi)\, (L_v g)$ (product rule)

3. Linear in v.

Derivations

The collection $\mathcal{D}_p = \{C^r(M) \to \mathbb{R} :\ (1) \text{ and } (2) \text{ hold}\}$ is called the derivations at p.

Algebraic Aside

$\delta_p : A \to \mathbb{R}$ given by $f \mapsto f(p)$ yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

Theorem

$T_pM \to \mathcal{D}_p$ given by $v = [\gamma] \mapsto L_v$ is a linear isomorphism.

Recall: Germ

Take $(f, U) \ni p$ and $(g, V) \ni p$. Then $(f, U) \sim (g, V)$ if and only iff $f \equiv g$ on $W \subset U \cap V$. The equivalence classes of this relation are germs.

Hadamard's Lemma

Take $f \in C^r(\mathbb{R}^n, 0)$ on $\mathbb{R}^n$ with $f(0) = 0$.
There exists $g_1, \ldots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$ such that

$$f(x) = \sum x_i g_i(x)$$

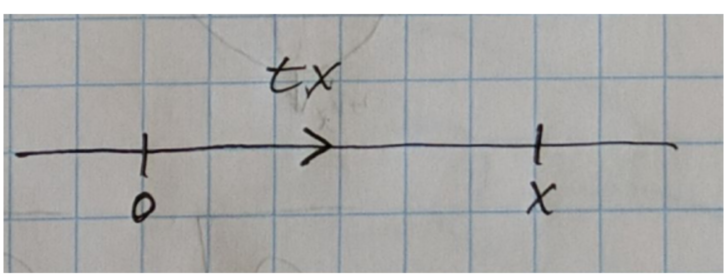
and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Example

For $n = 1$ and $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$ with $f(0) = 0$. Then we have $f(x) = x\overbrace{g(x)}^{c^{r-1}}$ given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

Proof of Example



Take

$$\int_0^1 \underbrace{\frac{d}{dt} f(tx)}_{f'(tx) \cdot x}\, dt = \underbrace{x \int_0^1 f'(tx)\, dt}_{g(x)}$$
$$= f(1 \cdot x) - f(0 \cdot x)$$

Proof of Lemma

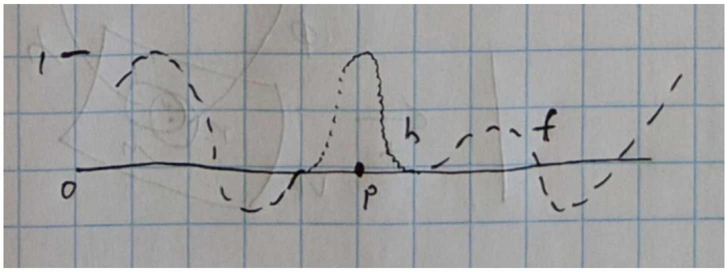
$$\int_0^1 \underbrace{\frac{d}{dt} f(tx)\, dt}_{\sum \frac{\partial f}{\partial x_i}(tx) \cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt}_{g_i}$$

Lemma

For $D \in \mathcal{D}_p$, Df depends only on the germ of f.

Proof

Need to show that if $f \equiv 0$ near p, $Df = 0$.



Where $\operatorname{supp} h \subset \{x : f(x) = 0\}$, $h(p) = 1$ and, consequently, $f \circ h = 0$.
So $D(0 = f \circ h)$, $D0 = Df \cdot h(p) + f(p)Dh$ and $0 = Df$.

Lemma

$D(k) = 0$ for k constant.

Proof

$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1)$, so $D(1) = 0$.
Arbitrary constants follow from the fact that D is linear.

Proof: One-to-One

Let $L_v f = L_w f$ for all $f = x_i$, then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each f and $v_i = w_i$.

Lemma

$$\operatorname{Der}(C^r(m)@p) = \operatorname{Der}(C^r(M, p))$$

Proof

Take $M = \mathbb{R}^n$ and $p = 0$.
Given D, we need $v = \sum v_i \frac{\partial}{\partial x_i}$ with

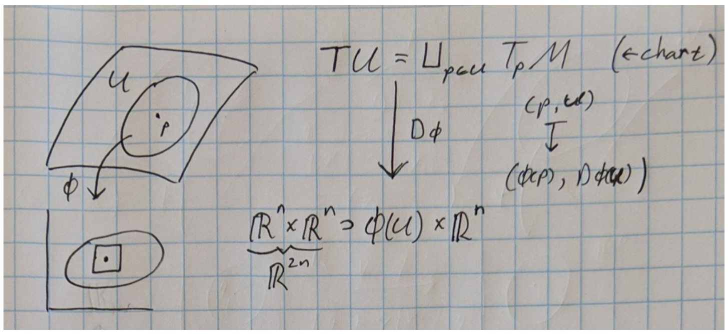
$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

for every f.
By Hadamard's lemma, write $f = \sum x_i g_i(x)$. Then

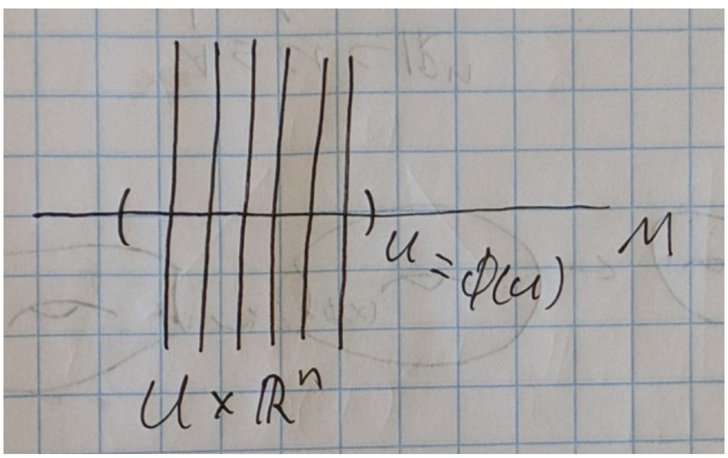
$$
\begin{aligned}
Df &= \sum (Dx_i) g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\
&= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0)
\end{aligned}
$$

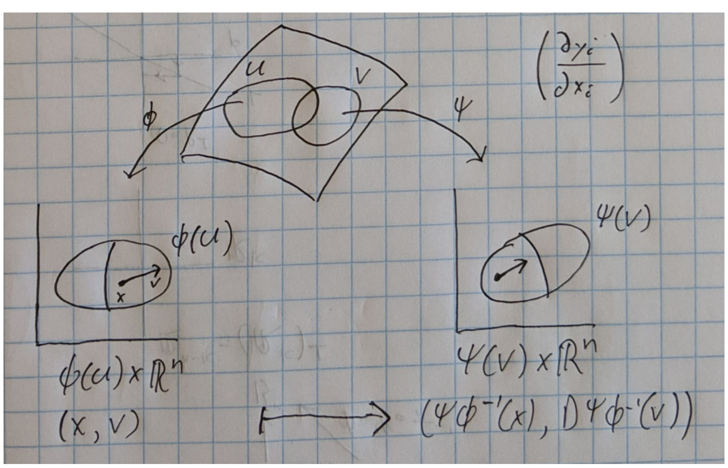
That is, since $v_i = L_v x_i$, we have $v_i = Dx_i$.

Definition: Tangent Bundle



For every $p \in M$, we have T_pM.
The tangent bundle $TM = \coprod_{p \in M} T_pM$.





When M is C^r, TM is C^{r-1}.

October 17, 2024

Chapter 7 of Lee (Lie Groups) will not be covered in class, but is highly recommended reading.

Preliminary Definition: Vector Fied

Take $M \overset{C^\infty}{\to} TM$ by $p \overset{v}{\mapsto} v(p) \in T_pM$.

Space of Vector Fields

Write $\mathfrak{X}(M)$ to be the collection of all C^∞ vector fields.

- This is a module over $C^\infty(M)$.
- $\mathfrak{X}(M)$ acts on $C^\infty(M)$ by $v \mapsto L_v$.

Smooth

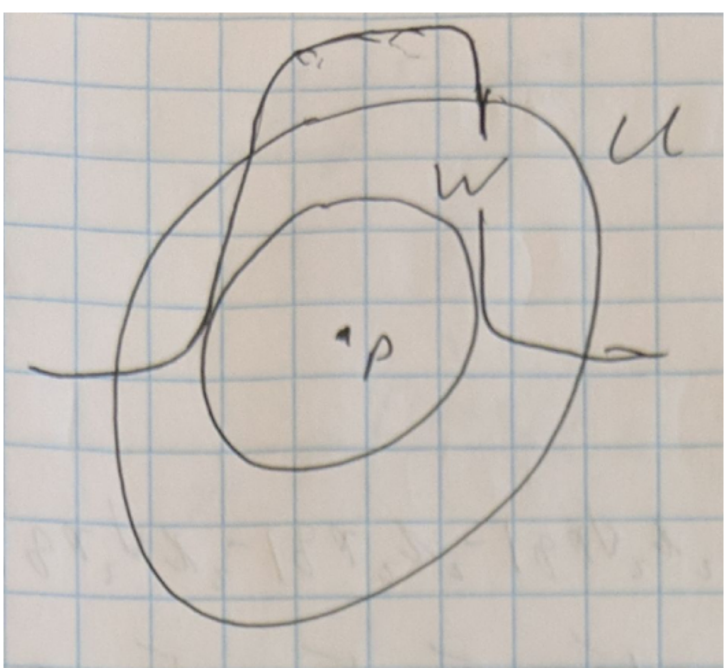
1. v is C^∞
2. In local coordinates, for $p \in U$ and $(U, \phi) = (x_1, \ldots, x_n)$ with $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ a basis in T_pM.

$$v = \sum \underbrace{v_i(x)}_{\substack{\text{functions} \\ \text{on } U}} \frac{\partial}{\partial x_i}$$

3. $f \in C^\infty(M) \implies L_v f \in C^\infty$

2 Implies 3

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i}$$



With $\phi|_W \equiv 1$, $\operatorname{supp}\phi = U$, $x_i\phi \in C^\infty(M)$, and $x_i\phi|W \equiv W_i$.
Then $L_v(x_i\phi) \underset{\text{on } W}{=} v_i$.

Definition: Lie Algebra

Take A a vector space equipped with (a Lie bracket) $[\cdot,\cdot] : A \times A \to A$ such that

- $[a,b] = -[b,a]$ (Skew Symmetric)
- $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ (Jacobi Identity)

Example 1

Take A to be an algebra and define $[a,b] = ab - ba$. Then if A is associative, it satisfies the Jacobi identity.

- $gl(n)$
- $so(n)$ (skew symmetric matrices)
    - $(ab-ba)^T = b^T a^T - a^T b^T = ba - ab = -[a,b]$
- $su(n)$ ($A^T = A^\dagger$)

Theorem:

The space of vector fields $\mathfrak{X}(M)$ is a Lie algebra:

- $\forall V, W \in \mathfrak{X}(M)$, $\exists! U \in \mathfrak{X}(M)$ such that $L_U f = L_W L_V f - L_V L_W f$. Write $U = [V,W]$.

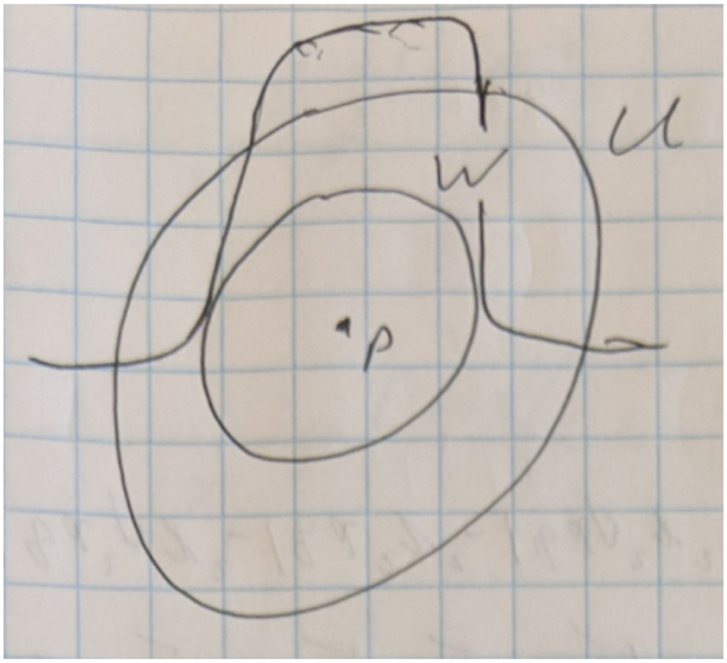
Lemma

$L_V f = L_W f$, $\forall f$ implies $V = W$.

Proof

$$V = \sum v_i \frac{\partial}{\partial x_i} \stackrel{?}{=} \sum w_i \frac{\partial}{\partial x_i} = W$$

Pick $p \in U$. We want to find $v_i(p) = w_i(p)$.



With $\phi|_W \equiv 1$, $\operatorname{supp}\phi \subset U$ and $f = x_i\phi \in C^\infty(M)$.

$$L_V f = L_W f$$
$$\sum v_j \frac{\partial\overbrace{(x_i\phi)}^{x_i}}{\underbrace{\partial x_j}_{\partial_{ij}}} = \sum w_j \frac{\partial\overbrace{(x_i\phi)}^{x_i}}{\underbrace{\partial x_j}_{\delta_{ij}}}$$

on W. Therefore $v_i = w_i$ on W.

Variant

For W_i an open cover, $L_V f = L_W f$ for all $f \in C^\infty(W_i)$ implies that $V = W$.

Compute

$$L_V L_W f = L_V\left(\sum w_j \frac{\partial f}{\partial x_j}\right) = \sum v_i \frac{\partial w_j}{x_i}\frac{\partial f}{\partial x_j} + \sum v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$L_W L_V f = \sum w_i \frac{\partial v_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum w_i v_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

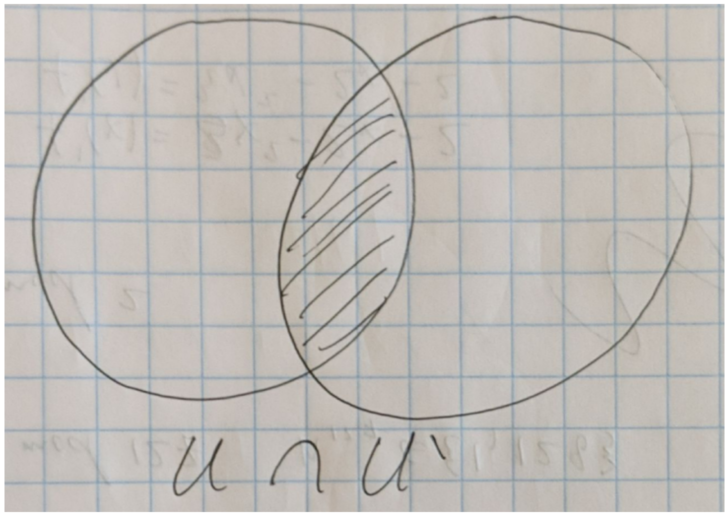
$$L_V L_W f - L_W L_V f = \sum_{i,j}\left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{x_i}\right)\frac{\partial f}{\partial x_j}$$

$$u = \sum_{i,j}\left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_i}{\partial x_i}\right)\frac{\partial}{\partial x_j}$$

Therefore $L_V L_W f - L_W L_V f = L_U f$.

Remark

Consider $U \ni u$ and $U' \ni u'$



Properties

- Lie algebra: Skew symmetric an dsatisfying the Jacobi identity
- Product rule: $[V, fW] = (L_V f)W + f[V, W]$.

Example

Let V be a finite dimensional vector space (e.g. $\mathbb{R}^n$).
Recall that $T_p V \cong V$ by $[p + tv] \leftarrow v$. Then $TV = V \times V\ (p, v)$.
Take $A \in \mathrm{End}(V)$ given by

$$v(x) = Ax = \sum v_i \frac{\partial}{\partial x_i} = \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Take also $w(x) = Bx$. Then $[V, W] = -(AB - BA)x = -[A, B]$.

Exercise

Given the example with W constant, determine $[V, W]$.

Theorem (Midterm Problem)

Take $A = C^\infty(M)$ and the derivations $D \in \mathrm{Der}(A)$ with $D : A \overset{\text{lin.}}{\to} A$ over $\mathbb{R}$ such that $D(fg) = Df \cdot g + f \cdot Dg$.
There exists a linear isomorphism $\mathfrak{X}(M) \overset{\cong}{\to} \mathrm{Der}(A)$ given by $v \mapsto L_v$.

Lemma

Take $D \in \mathrm{Der}(A)$. Then $D_p \in \mathcal{D}_p$ where $D_p f := (Df)(p)$.

$$D_p(fg) = (D_p f)g(p) + f(p)(D_p g)$$

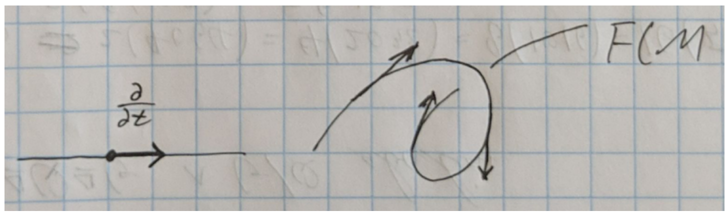
We know from above that $\mathcal{D}_p \cong T_p m$. Therefore $L_v f = Df$.

Definition: Push Forward

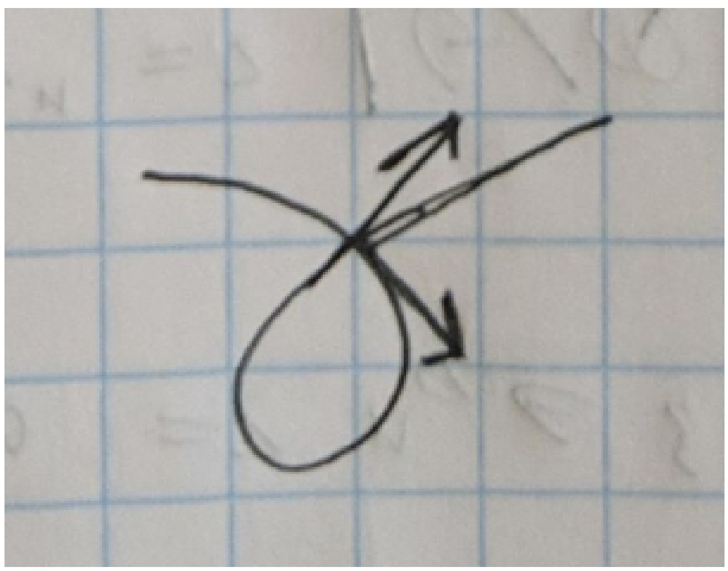
Take $F : M \to N$ which gives rise to $F_* : T_p M \to T_{F(p)} N$ (equivalent to $F_* : \mathcal{D}_p \to \mathcal{D}_{F(p)}$) given by $[\gamma] \mapsto [F \circ \gamma]$.
We see that $(F_* D)(g) := D(g \circ F)$.

You Cannot Push Forward Vector Fields

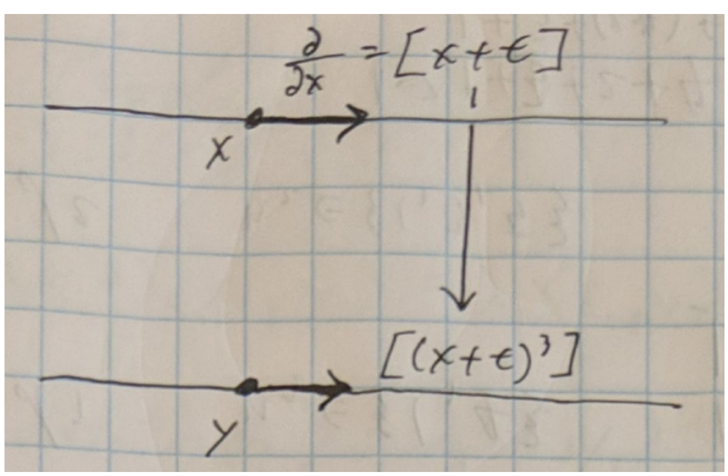
- if F is not surjective



- if F is not injective



- it is possible that $F_* V$ would fail to be smooth. Take $(F_* v)(y) = 3y^{2/3} \frac{\partial}{\partial y}$



Remark

Take a diffeomorphism $F : M \to N$.
Then $F : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is given by $F_*[V, W] = [F_* V, F_* W]$.

October 22, 2024

Dimension of Manifolds

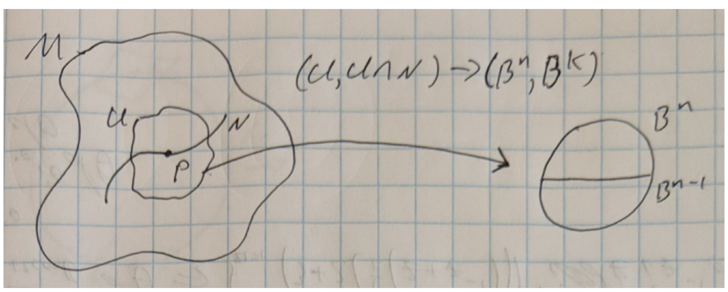
Take connected manifolds M^m and N^n with diffeomorphism $M^m \underset{G}{\overset{F}{\leftrightarrows}} N^n$. If $p \in M^n$ and $q = F(p)$, then

$$T_p M \underset{DG}{\overset{DF}{\leftrightarrows}} T_q N$$

is linear and $\dim T_p M = \dim T_q N$.

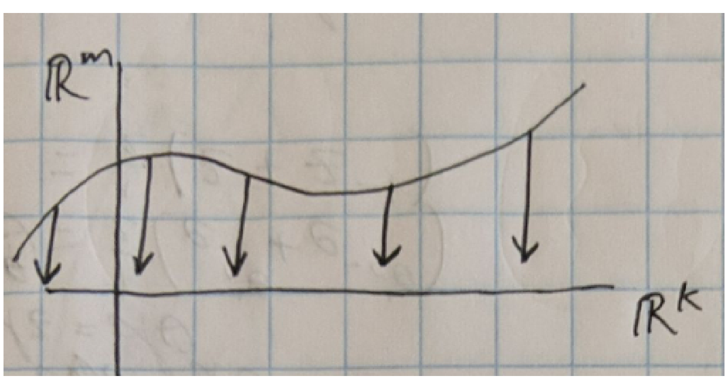
Definition: Submanifold

$N \subseteq M^n$ is a submanifold if $\forall\, p \in N$, there exists a neighborhood $U \ni p$ in M and a diffeomorphism $U \to B^n$ which satisfies $U \cap N \to B^k = B^n \cap \mathbb{R}^k$.



Example 1

Consider $F : \mathbb{R}^k \to \mathbb{R}^n$ and its graph $\mathrm{Graph}(F) = \{(x, F(x)\}$. This is a submanifold in $\mathbb{R}^k \times \mathbb{R}^n$.



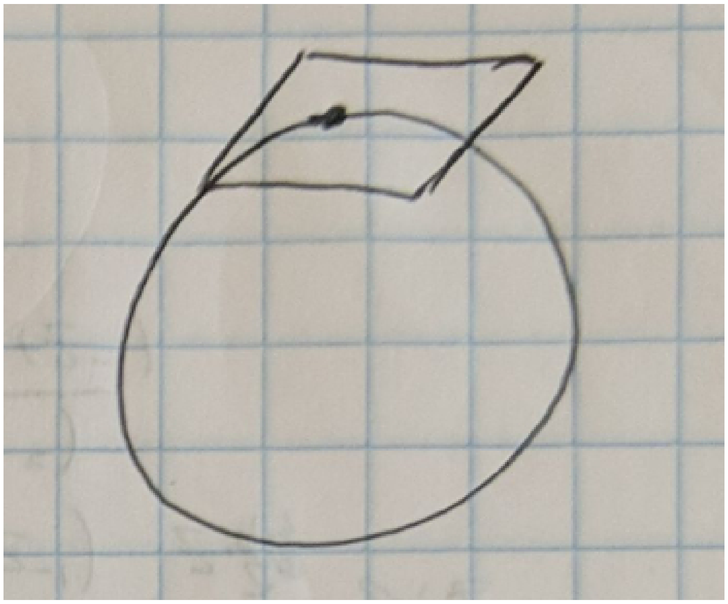
$\mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k \to \mathbb{R}^m$ given by $(x, y) \mapsto (x, y - f(x))$.

Example 2

Take $F : X \to Y$, X, Y manifolds. Then $\mathrm{Graph}(F) \subset X \times Y$ is a submanifold.

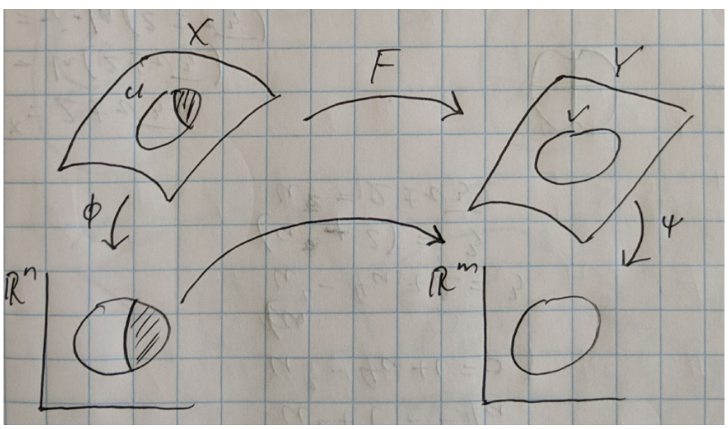
Example 3

$S^1 \subset \mathbb{R}^2$ or $S^{n-1} \subset \mathbb{R}^n$



Example 4

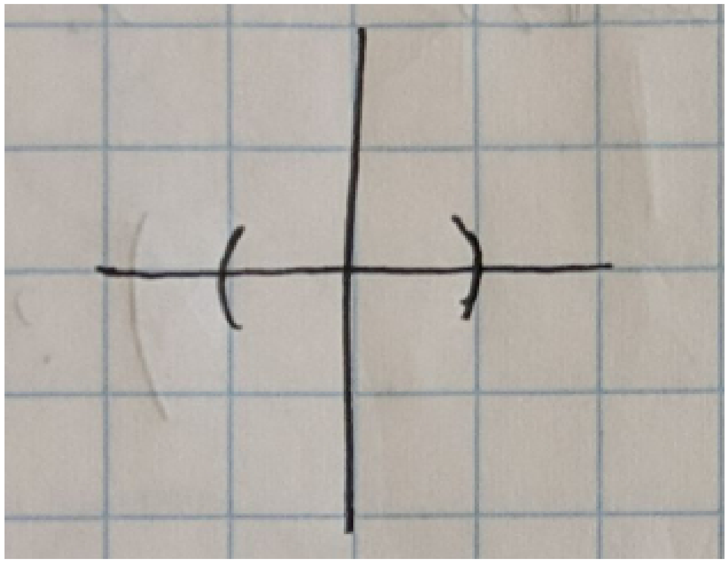
A graph is always a submanifold.



$\mathrm{Graph}\left(F|_{U \cap F^{-1}(v)}\right) \subset U \times V$ implies $\mathrm{Graph}\left(\psi F p^{-1}|\ldots\right) \subset \phi(U) \times \psi(V)$.

Example 5

$(-1,1) \times 0 \in \mathbb{R}^2$.



Example 6

$\mathbb{RP}^n \subseteq \mathbb{CP}^n$.

Definition: Regular Point

Let M, N be manifolds with $p \in M$, $q = F(p) \in N$ and take $F : M \to N$ with $DF : T_pM \to T_qN$.
p is a regular point if DF is onto; q is a regular value if each $p \in F^{-1}(q)$ is a regular point.
Otherwise, they are called critical points or critical values.

Remark

If $\dim M < \dim N$, then all points in M are critical and all values in $F(M)$ are critical (while $N \setminus F(M)$ are regular).

Remark

For $f : M \to \mathbb{R}$ with $df : T_pM \to \mathbb{R} \cong T_{f(p)}\mathbb{R}$. Locally, $df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = \nabla f$. Then p is regular if and only if $df \neq 0$.
Equivalently, p is regular if and only if there exists $v \in T^pM$ such that $L_v f \neq 0$.

Proof

$L_v f = df(v) = \sum \frac{\partial f}{\partial x_i} v_i$.

Theorem:

Let q be a regular value. Then $F^{-1}(q)$ is a submanifold.

Example 1

$f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \sum \lambda_i x_i^2$, $\lambda_i \neq 0$. Then $f^{-1}(1)$ is a submanifold.

Proof

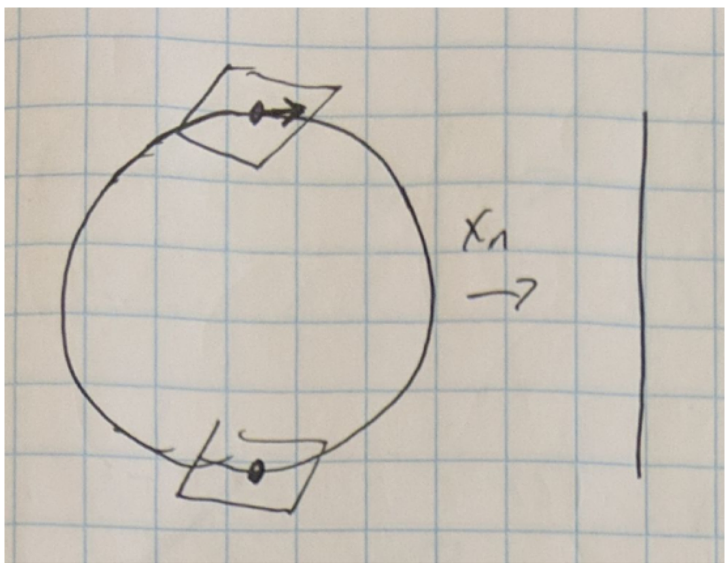
x is a regular point when $f(x) \neq 0$ ($\nabla f \neq 0$).

$$v = \sum x_i \frac{\partial}{\partial x_i} = x$$

$$L_v f = \sum 2 \underbrace{\lambda_i x_i}_{\frac{\partial f}{\partial x_i}} \underbrace{x_i}_{v_i} = 2f(x) \neq 0$$

Example 2

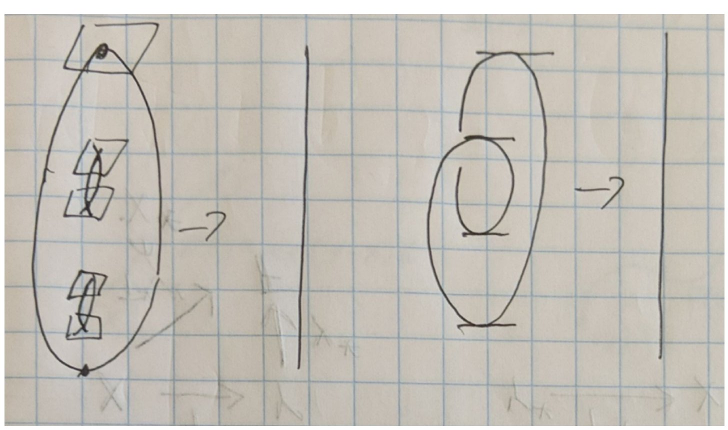
$S^{n-1} \subset \mathbb{R}$



$f = x_n : S^{n-1} \to \mathbb{R}$ the height function, so

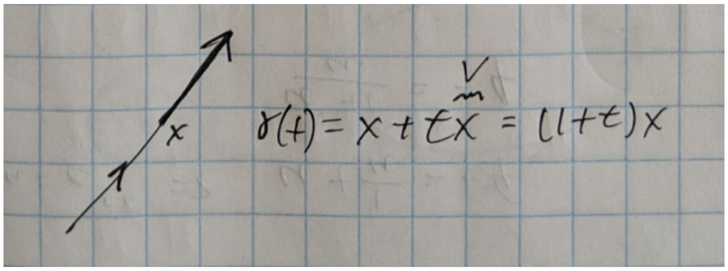
$$L_v f = L_v x_n = v_1 \frac{\partial x_n}{\partial x_1} + \cdots + v_{n-1} \frac{\partial x_n}{\partial x_{n-1}} + \underbrace{v_n}_{=0} \frac{\partial x_n}{\partial x_n}$$

The same follows for all projective maps.



Theorem:

$f(\lambda x) = \lambda^k f(x)$



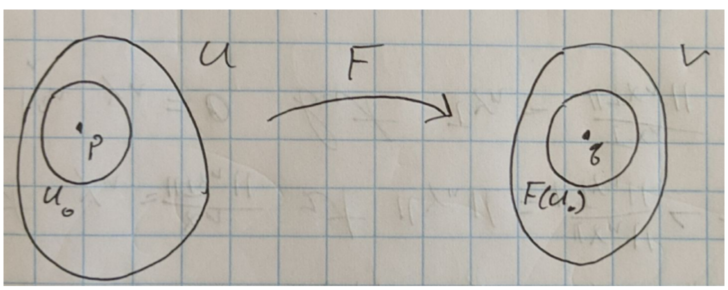
$$
\begin{aligned}
L_v f &= \frac{d}{dt} f((1+t)x)|_{t=0} \\
&= \underbrace{\frac{d}{dt}(1+t)^k}_{k}|_{t=0} f(x)
\end{aligned}
$$

Theorem: Inverse Function Theorem

Take $U, V \subseteq \mathbb{R}^n$ open and a map $F : U \to V$ with $p \in U$ and $q = f(p) \in V$. Then take

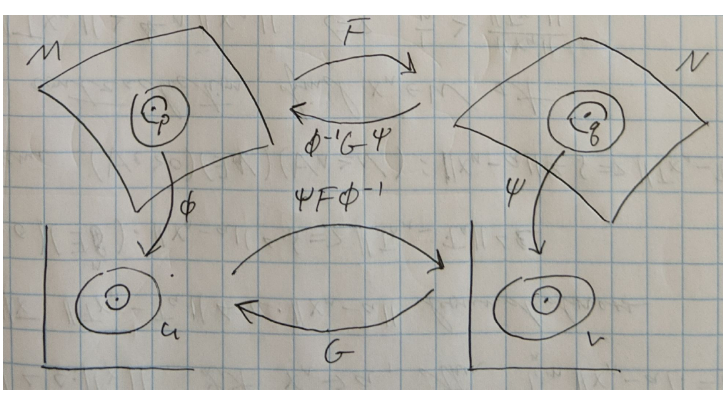
$$\{DF : T_p U \to T_q V\} = \frac{\partial F_i}{\partial x_j}$$

such that $\mathrm{Rank}\, DF = n$ is an isomorphism.
Then there exists $U_0 \ni p$ such that $U_0 \xrightarrow{F} F(U_0)$ is a diffeomorphism.



Corollary

If $F : M^n \to N^n$ with $DF : T_p M \xrightarrow{\cong} T_q N$ ($\mathrm{Rank}\, DF = n$).
Then there exists a neighborhood $U_0 \ni p$ such that $F : U_0 \to F(U_0)$ is a diffeomorphism.



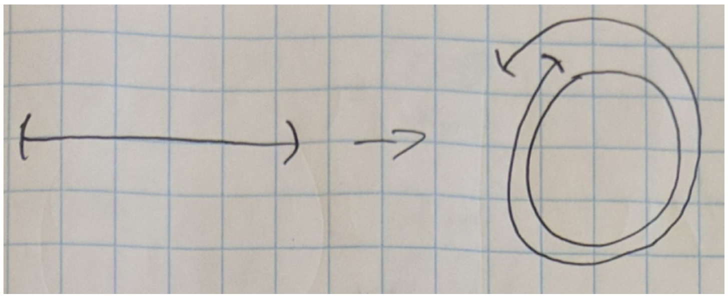
$D(\psi F\phi^{-1}) = D\psi D f D\phi^{-1}$.

Local Diffeomorphism

Diffeomorphisms are local but local diffeomorphisms are not necessarily diffeomorphisms.

Example 1

Take $\mathbb{R}$ or any interval longer than 1 and map them to S^1 by $t \mapsto e^{2\pi i t}$.



Example 2

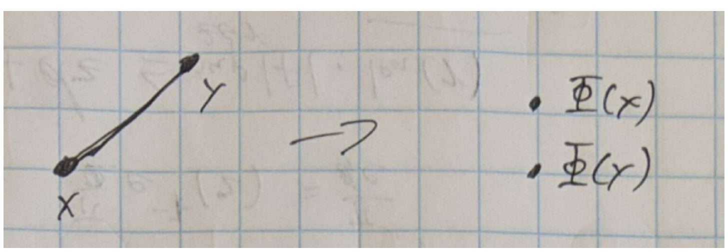
$S^1 \to S^1$ given by $z \mapsto z$

Example 3

$S^n \to \mathbb{RP}^n$.

Contraction Mapping Principle

Take $\Phi : X \overset{C^0}{\to} X$ with X complete such that $d(\Phi(x), \Phi(y)) < c \cdot d(x, y)$ with $c \in (0, 1)$.
Then there exists a unique fixed point $\Phi(p) = p$.



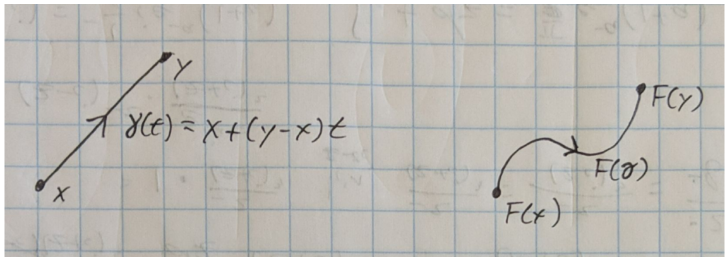
Example

Take $F : \mathbb{R}^n \to \mathbb{R}^n$ and $||DF_x|| < c < 1$.
Recall that the operator norm for an operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$||A||_{\text{op}} = \sup_{v \neq 0} \frac{||Av||}{||v||} = \sup_{||v||=1} ||Av||$$

Then F is a contraction by c.

Proof



$$d(x,y) = ||y-x|| = \int_0^1 ||\gamma'(t)||\,dt$$

Then

$$\begin{aligned}
d(F(x),F(y)) &\leq \int_0^1 ||F'(\gamma(t))||\,dt \\
&\leq \int_0^1 ||DF_{\gamma(t)} \cdot \gamma'(t)||\,dt \\
&\leq \int_0^1 \overbrace{||DF_\gamma(t)||}^{<c} \cdot ||\gamma'(t)||\,dt \\
&\leq c \cdot \int_0^1 ||\gamma'(t)||\,dt \\
&\leq c \cdot d(x,y)
\end{aligned}$$

Proof

Write $\{x_k\} = \{\Phi^k(x)\}$.
Claim: $\{x_k\}$ is Cauchy which implies that $x = \lim_{n\to\infty} x_n$. Then

$$\Phi(x) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

Uniqueness follows from the observation that for fixed points $d(x,y) \leq d(x,y)$ implies $x = y$.

Proof that Sequence is Cauchy

For $n \leq n+k$,

$$d(x_n, x_{n+k}) \leq d(\Phi(x_{n-1}, \Phi(x_{n+k-1})) \leq cd(x_{n-1}, x_{n+k-1}) \leq c^n d(x_0, x_k) \leq c^n L \cdot (1 + \cdots + c^{k-1}) \leq \frac{L}{1-c} c^n \underset{n\to\infty}{\to} 0$$

since

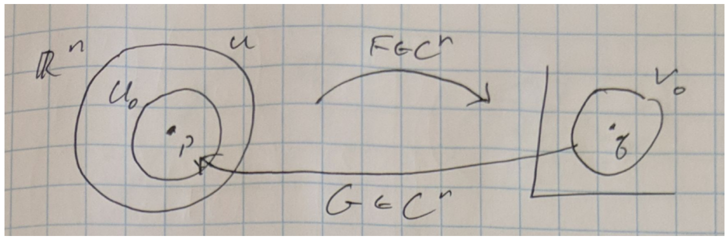
$$d(x_0, x_k) \leq \underbrace{d(x_0, x_1)}_{L} + \underbrace{d(\overbrace{x_1, x_2}^{\Phi(x_0),\Phi(x_1)})}_{c\cdot L} + \cdots + \underbrace{d(x_{k-1}, x_k)}_{c^{k-1}\cdot L}$$

October 24, 2024

Recall: Contraction Mapping Theorem

$\Phi : X \to X$ complete, then there exists a unique fixed point x such that $\Phi(x) = x$.

Recall: Inverse Function Theorem



$DF_P : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$ implies there exist U_0 and V_0 such that

$$F : U_0 \underset{G}{\leftrightarrows} V_0$$

is a diffeomorphism with $F \circ G = I$ and $G \circ F = I$.

Remarks

Assume that $p = 0 = q$ and that $DF_0 = I$.
Then $F = I + \phi$ and $D_{\phi_0} = 0$ (contracting). Look for $G = I + g$, then

$$\begin{aligned} F \circ G &= I \\ (I + \phi) \circ (I + g) &= I \\ -\phi \circ (I + g) &= g \end{aligned}$$

which gives us our fixed point. Then $\Phi : g \mapsto -\phi \circ (I + g)$ with $g : \overline{B(r)} \to \mathbb{R}^n$ is a continuous function on a Banach space giving

$$\underbrace{\overline{B(r)}}_{r \text{ adjustable}} \xrightarrow{C^0} \underbrace{B(R)}_{R \text{ fixed}}$$

on a subset with $||g|| = \sup_{x \in B(r)} g(x)$ and $||g|| < R$.

Claim

Φ is contracting.
The contraction mapping theorem implies there exists a unique fixed point $\Phi(g) = g$ where $-\phi \circ (I + g) = g$.

Observations

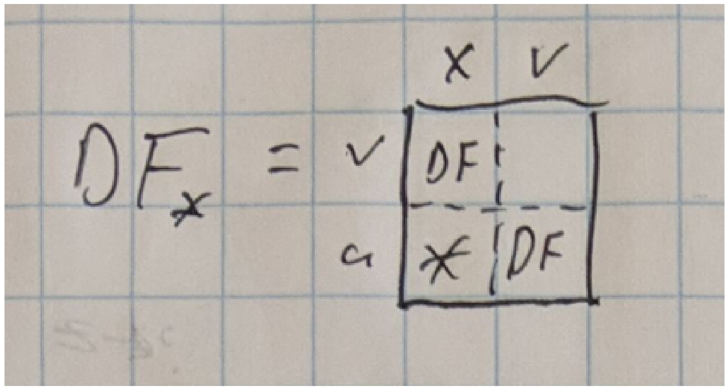
F is C^1 and DF invertible implies G is C^1.
$F \circ G = I$, $F(G(x)) = x$, $(F' \circ G) \circ G' = I$.
So $G' = (F' \circ G)^{-1}$.

Remark

$$\underbrace{U \times \mathbb{R}^n}_{TU} \underset{(F, DF)}{\overset{F_*}{\to}} \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{T\mathbb{R}^n}$$

$$\underset{(G, DG)}{\overset{G_*}{\leftarrow}}$$

Then F being C^2 implies F_* is C^1 and invertible, and G_* is C^1.
$F_*(x,v) = (F(x), DF_x v)$

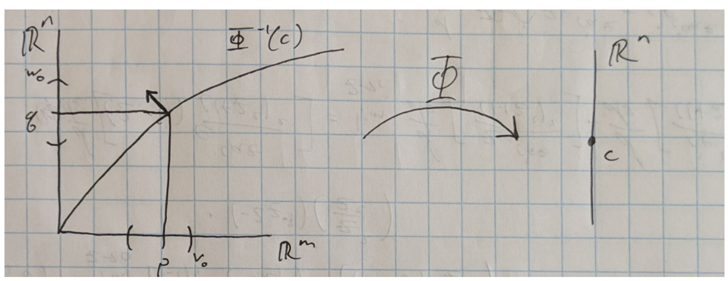


Theorem: Implicit Function Theorem

Take

$$(x,y) \in \mathbb{R}^m \times \mathbb{R}^n \supset U \ni (p,q) \xrightarrow[C^r]{\Phi} \mathbb{R}^n \ni c$$

with $\frac{\partial \Phi}{\partial y}$ an invertible $n \times n$ matrix.



$(\nabla \Phi = \left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \Phi}{\partial x_n}, \frac{\partial \Phi}{\partial y} \neq 0\right))$.
Then there exist $V_0 \ni p$ and $W_0 \ni q$ such that in $V_0 \times W_0$ $\Phi^{-1}(c)$ is the graph of $g : V_0 \overset{C^r}{\to} W_0$ such that $\Phi(x, g(x)) = c$.

Remark

IFT $\iff$ IFT.

Proof that Inverse Implies Implicit

Define $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $(x,y) \mapsto (x, \Phi(x,y))$ (i.e. $(p,q) \mapsto (p,c)$).

$$DF = \begin{array}{cc} & \begin{array}{cc} \mathbb{R}^m & \mathbb{R}^n \end{array} \\ \begin{array}{c} \mathbb{R}^m \\ \mathbb{R}^n \end{array} & \left[\begin{array}{c|c} I & * \\ \hline 0 & \frac{\partial \Phi}{\partial y} \end{array}\right] \end{array}$$

invertible. Applying the inverse function theorem, $F \circ G = I$ implies $G = (\mathrm{id}, g)$ with $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ where $g(\cdot) = g(\cdot, c)$. So $\Phi(x, g(x)) = c$.

Theorem: Regular Value Theorem

Take $F : M^{n+m} \to N^n \ni c$ with c a regular value.
Then $F^{-1}(c)$ is a submanifold.

Proof

We know that each $p \in F^{-1}(c)$ is a regular point.
Then in local coordinates $DF_p : T_p M^{n+m} \to T_c N^n$ is given by an $n \times (m+n)$ matrix of rank n.

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ c & \left[\begin{array}{c|c} \qquad & \frac{\partial F}{\partial y} \end{array}\right] \end{array}$$

then locally F^{-1} is a graph and therefore a submanifold.

Definition: Immersion

A map $F : N^n \to M^m$ is an imersion if $DF_p : T_p N \to T_{f(p)} M$ is one to one at every point ($m \geq n$).

Definition: Embedding

The map $F : N^n \to M^m$ is an embedding if it is an immersion and a homeomorphism on its image.

Definition: Submiersion

The map $F : N^n \to M^m$ is a submersion if DF_p is onto ($m \le n$).

Examples

Example 1

A local diffeomorphism is both an immersion and a submersion.

Example 2

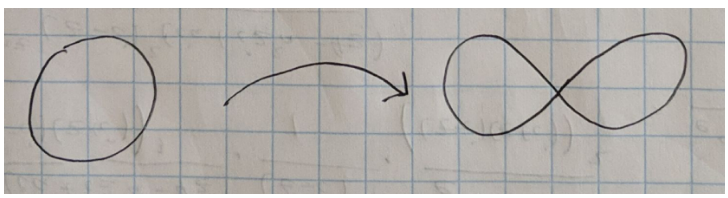
$\gamma : \mathbb{R} \to M$ is an immersion if and only if $\gamma'(t) \neq 0$ for all points t.

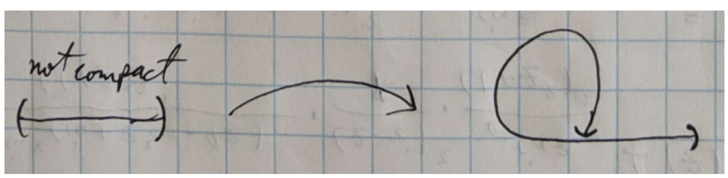
Example 3

For N compact, F is an embedding if and only if it is one-to-one and an immersion.

Counter-example 1

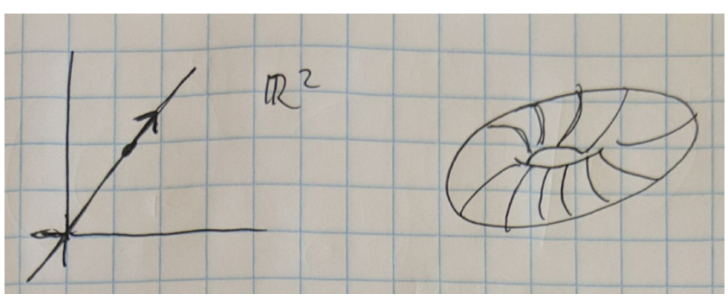
Imersions but not embeddings.





Counter-example 2

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$$
$$t \mapsto (t, at)$$



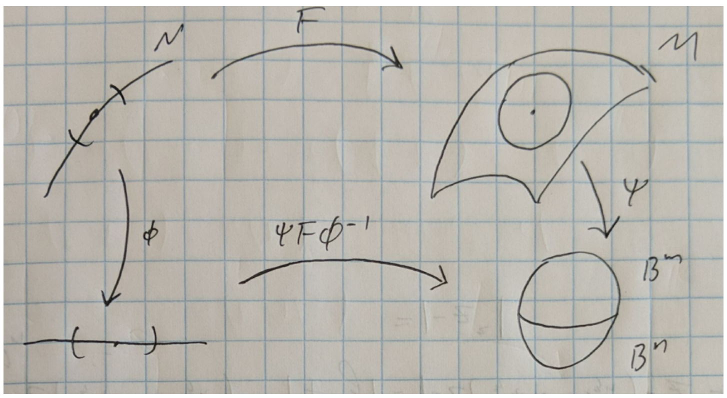
Can be given by composition of $(e^{2\pi i t}, e^{2\pi i a t})$.

Remark

Consider $A : V \to V$ with equivalence given by $BAB^{-1} \sim A$.
If $A : V \to W$ then the equivalence given by $B_1 A B_2^{-1} \sim A$.
Consider $f(\mathbb{R}, 0) \to \mathbb{R}$ with the assumption that $f(0) = 0$, $f'(0) = 1$. Then there exists a change of coordinates $x = x(t)$ such that $f(x(t)) = t$.
By the inverse function theorem, there exists g satisfying $f(g(t)) = t$.
Consider $f(z)$ complex analytic with $f(0) = 0$ and $f'(0) = 1$.

Local Normal

Let $F : p \in N^n \to M^m \ni q$ be an immersion.
Then there exist local coordinates $x_1, \ldots, x_n$ near p and $y_1, \ldots, y_n$ near q such that $y_1 = x_1, \ldots, y_n = x_n, y_{n+1} = 0, \ldots$.



October 29, 2024

Recall: Local Normal Form Theorem for Immersions

For $F : N^n \to M^m$ an immersion where $\operatorname{rank}(F) = n \leq m$ and $p \in N$, there exists a coordinates $x_1, \ldots, x_n$ near p and $y_1, \ldots, y_m$ near $F(p)$ such that

$$F : \begin{array}{c} y_1 = x_1 \\ \vdots \\ y_n = x_n \\ y_{n+1} = 0 \\ \vdots \\ y_m = 0 \end{array}$$

IMAGE 1

Proof

$$p \in V \xrightarrow{F} U \ni q$$
$$p \to q \in F(p)$$

$\operatorname{rank} DF_p = n \leq m$, for example.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

IMAGE 2

$$\begin{matrix} y_1 = z_1 \\ \vdots \\ y_n = z_n \end{matrix} \quad \text{and} \quad \begin{matrix} y_1 = x_1 \\ \vdots \\ y_n = x_n \\ y_{n+1} = z_{n+1} - F_{n+1}(z_1, \dots, z_n) \\ \vdots \\ y_m = z_m - F_m(z_1, \dots, z_m) \end{matrix}$$

Theorem: Local Normal Form Theorem for Submersions.

Assume that $F : N^n \to M^m$ is a submersion such that $\operatorname{rank} DF = m \leq n$.
Then we may take coordinates $x_1, \dots, x_n$ and $y_1, \dots, y_m$ such that $y_1 = x_1, \dots, y_m = x_m$.
The procedes similarly to previous theorem.

Remark

Immersions look locally like linear subspaces.
Submersions look locally like projections.

Theorem: Constant Rank Theorem

Assume that $\operatorname{rank} DF = k$ for a constant k.
Then for coordinates $x_1, \dots, x_n$ and $y_1, \dots, y_m$, we can have $y_1 = x_1, \dots, y_k = x_k$, $y_{k+1} = 0, \dots$

Homework

Take f a germ at 0 on $\mathbb{R}^n$ and take $df_0 \neq 0$ ($\nabla f_0 \neq 0$).
Then there exists a coordinate system $y_1, \dots, y_n$ such that $f(y) = f(0) + y_1$.
That is, for $(x_1, \dots, x_n)$ there exists a diffeomorphism $(f \circ \phi)(x_1, \dots, x_n) = f(0) + x_1$.
Note that we do not have control over the x coordinates, so the previous theorems will not work.

Remark

Take $K \xrightarrow{G} N \xrightarrow{F} M$.
If G, F are submersions, then FG is a submersion.
If G, F are immersions, then FG is an immersion.

Theorem

Let $F : N \to M$ be an embedding.
Then $F(N)$ is a submanifold. Consequently, $F : N \to F(N)$ is a diffeomorphism.

IMAGE 3

If N is a submanifold of M, then $N \hookrightarrow M$ is an embedding.

Example

Take $\mathbb{R} \to \mathbb{R}^2$ by $t \mapsto (0, t^3)$.

IMAGE 4

This is not even an immersion.

Example 2 (Homework)

For N the graph of $y = |x|$, there exists $F : \mathbb{R} \underset{1-1}{\overset{C^\infty}{\to}} \mathbb{R}^2$ such that $N = F(\mathbb{R})$.

IMAGE 5

Submanifolds

For any $q \in X$, there exists a neighborhood U and a diffeomorphism $\phi : (U, U \cap X) \to (W, W \cap \mathbb{R}^n)$.

IMAGE 6

Proof

And embedding is an immersion which is homeomorphic on $F(N)$
With F an immersion, we have

IMAGE 7

Since F is also a homeomorphism, there exists U' such that $U' \cap X = F(V)$.

Zero Measure Sets

Key Point

Zero measure sets are unambiguously defined.

Definition: Zero Measure Set

$X \subseteq M$ is a zero measure set if

- For every closed chart (U, ϕ), $\phi(U \cap X) \subset \mathbb{R}^n$ has zero Lebesgue measure.
- There exists a coordinate atlast $(_i, \phi_i)$ such that $\phi_i(U_i \cap X)$ has zero measure.

Lemma

$\mathbb{R}^n \supset W \underset{C^1}{\overset{F}{\to}} \mathbb{R}^n$ and $Y \in W$ where $m(Y) = 0$.
Then $F(Y)$ has zero measure.

Proof

IMAGE 8

We may cover Y by a countable collection of open sets U such that $\overline{U} \subset W$.
Then it suffices to show taht $F(Y \cap U)$ has zero measure.
Then $||DF|| \leq C$. Given $\varepsilon > 0$, we need a cover $F(Y)$ by open balls with total volume less than ε.
Pick $p \in Y$ and define $B_r(p) = \{z \in U : ||z - p|| \leq r\}$. Then $F(B_r(p)) \subset B_{Cr}(F(p))$.

IMAGE 9

For $p = 0 = F(p)$

IMAGE 10

$$||F(z)|| \leq \text{length}(t \mapsto F(tz)) = \int_0^1 \left|\left|\frac{d}{dt}F(tz)\right|\right| \, dt \leq \int_0^1 ||DF|| \cdot ||z|| \, dt \leq Cr$$

Since Y has zero measure, it can be covered by balls $B_{r_i}(p_i)$ with total volume less than δ.
Therefore $F(Y)$ is covered by $B_{Cr_i}(F(p_i))$. Then the sum is bounded above by $C^n\delta$.
Take $\delta = \frac{\varepsilon}{C^n}$.

Definition: Smooth Measure

Let μ be a measure on M.
μ is smooth if

- for every coordinate chart $(U, \phi = (x_1, \ldots, x_n))$, $\mu = f \, dx_1 \ldots dx_n$ $(f > 0)$.
- there exists a coordinate atlas (U_i, ϕ_i) such that the same is true.

Proposition

A smooth measure exists.

Proof

Take (U_i, ϕ_i), $dx_1 \ldots dx_n = \mu_i$, and the partition of unity h_i subordianted to U_i ($\operatorname{supp} h_i \subseteq U_i$ and $\sum h_i \equiv 1$)
Then $h_i\mu_i$ is a measure on M. Then we may take $\sum h_i\mu_i$.

Proposition

The following three conditions are equivalent.

- X is a zero measure set
- there exists a smooth, positive μ such that $\mu(X) = 0$
- every smooth positive μ has $\mu(X) = 0$.

October 31, 2024

Theorem: Sard

Let $F : M^m \overset{C^r}{\to} N^n$ with $r > \max\{0, m-n\}$. The set of critical values has zero measure.

Remarks

There are two cases. $m \leq n$ permits C^1. This is the easy case because there are no C^1 space filling curves.
If $m > n$, then $r > m - n + 1$. This case is difficult.

Corollary

There exist regular values.
The set of regular values is everywhere dense.

Proof

For $M = [0,1]$, $N = \mathbb{R}$, $n = m = 1$, and $f : [0,1] \overset{C^r}{\to} \mathbb{R}$.
Separate the interval into segments I_i of length $1/k$.

IMAGE 1

Consider all intervals containing critical points.
Then f' is uniformly continuous. That is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \implies |f'(x) - f'(y)| < \varepsilon$.
Consider the critical values covered by intervals of total length $< \varepsilon$ and $f(I_i)$ containing critical points.
Choose $1/k < \delta$, then $|f'|_{I_i}| < \varepsilon$. So $|f(I_i)|\varepsilon \cdot 1/k$ and, consequently,

$$\sum_{I_i} |f(I_i)| < k \cdot \frac{k}{\varepsilon} = \varepsilon$$

Theorem: Brouwer Fixed Point Theorem

A map $G : \overline{B}^n \overset{C^0}{\to} \overline{B}^n$ has a fixed point $G(x) = x$.

Fact

The one dimensional manifolds are exactly $\mathbb{R} = (0,1)$, S^1, $[0,1)$ and $[0,1]$.

Fact

Consider $F : M \to N$ where $\partial N = \varnothing$ with q regular for F and $F|_{\partial M}$.
Then $F^{-1}(q)$ is a manifold with boundary and $\partial F^{-1}(q) = F^{-1}(q) \cap \partial M$.

IMAGE 2

Lemma

Let M be a compact manifold with boundary. Then no map exists such that $F : M \overset{C^\infty}{\to} \partial M$ such that $F|_{\partial M} = \text{id}$.

Proof

Write

$$\begin{tikzcd} F : M \arrow[r] & \partial M \ni q \\ \partial M \arrow[u, hook] \arrow[ur, "id"] & \end{tikzcd}$$

where q is a regular value.
Then $F^{-1}(q)$ is a submanifold with dimension $\dim M - \dim \partial M = n - (n-1) = 1$.
Then the only candidate closed manifold with boundary of dimension 1 is $[0, 1]$.

IMAGE 3

But since the restriction is the identity and the boundary must lie in the boundary of M, we have a contradiction.

Corollary

There is no function $F : \overline{B}^n \overset{C^\infty}{\to} S^{n-1}$ such that $F|_{\partial \overline{B}^n = S^{n-1}} = \mathrm{id}.$

Lemma

$G : \overline{B}^n \overset{C^\infty}{\to} \overline{B}^n$ has a fixed point.

Proof

Assume the contrary, that there exists $g : \overline{B}^n \overset{C^\infty}{\to} \overline{B}^n$ such that $g(x) \neq x$.

IMAGE 4

Then consider $x \mapsto G(x)$ with $G|_{\partial \overline{B}^n} = \mathrm{id}$. Then, explicitly,

$$g(x) + t \cdot \frac{x - g(x)}{||x - g(x)||} = G(x)$$

However, by the previous lemma, this is impossible.

Fact

Every continuous map $G : \overline{B}^n \to \mathbb{R}^k$ can be C^0-approximated by a smooth map P.

$$||G - P|| = \sup_{x \in \overline{B}^n} ||G(x) - P(x)|| < \mu$$

Proof of Theorem

Assume $G : \overline{B}^N \to \overline{B}^n$ with $G(x) \neq x$ with $0 < \varepsilon < \min ||G(x) - x||$.
However $P : \overline{B}^n \to \overline{B}^n(1 + \varepsilon)$. Then for $F = \frac{1}{1+\varepsilon} P$, $G \approx P \approx F$.

Theorem

Let M^n a smooth manifold. Then there exists an embedding $M \hookrightarrow \mathbb{R}^N$.

Proof

(Assuming that M is compact)
Pick some finite cover by coordinate charts (U_i, ϕ_i) such that $V_i \subseteq \overline{V}_i \subseteq U_i$ with V_i also a finite cover.

IMAGE 5

Let f_i be such that $f_i|_{V_i} \equiv 1$ and $\operatorname{supp} f_i \subset U_i$. Further, assume that $f < 1$ outside $\overline{V}_i$.

IMAGE 6

Then $f_i\phi_i : M \to \mathbb{R}^n$.
Define $F : M \to \mathbb{R}^n$ by $(f_1\phi_1, \ldots, f_i\phi_k, f_1, \ldots, f_k)$ such that $N = nk + k$.
Claim: F is an embedding. That is, it is an immersion which is homeomorphic on its image. Equivalently, it is an immersion and one-to-one.
We need that $\operatorname{rank} DF = n$. Take $x \in V_1$ where $f_1\phi_1 = \phi_1$ on V_1. Then $D(f_1\phi_1)_x = (D\phi_1)x$ which is of rank n. So F must be of at least (and, in fact, at most) rank n. That is, F is an immersion.
Again, take $x \in V_1$ and consider $F(x) = F(y)$. Then $f_1(x) = f_1(y)$ and, consequently, $y \in \overline{V}_1$.
Therefore $f_1\phi_1(x) = f_1\phi_1(y)$ which implies $\phi_1(x) = \phi_1(y)$ and $x = y$. Therefore F is one-to-one.

November 5, 2024

Theorem: (Weak) Whitney Embedding Theorem

If M^n is a smooth manifold, then

- there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.
- there exists an immersion $M \hookrightarrow \mathbb{R}^{2n}$.

Remark: (Strong) Whitney Embedding Theorem

For M^n smooth and compact,

- there exists an embedding $M \hookrightarrow \mathbb{R}^{2n}$
- there exists an immersion $M \hookrightarrow \mathbb{R}^{2n-1}$

This is sharp, however we can do better for some M (e.g. $S^n \hookrightarrow \mathbb{R}^{n+1}$.

Tangent Bundle

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ U \subset M \ni p \end{tikzcd} \qquad T_pM = \pi^{-1}(p)$$

IMAGE 1

Where we have a basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ in T_pM for any $p \in U$. Then, writing $v = \sum v_i \frac{\partial}{\partial x_i}$,

$$TM|_U = \pi^{-1}(U) \underset{C^{r-1}}{\overset{(\phi, D\phi)}{\mapsto}} (\underbrace{\phi(p)}_{\phi\pi(p)}, \underbrace{(v_1, \ldots, v_n)}_{D\phi(v) \in \mathbb{R}^n})$$

Properties

$F : M \xrightarrow{C^r} N$ implies $DF : TM \xrightarrow{C^{r-1}} TN$ by $v \mapsto DF(v)$ where $DF(V)$ is $F_* v$.
If F is an embedding (or immersion), then so is DF.
If F is a submersion, then so is DF.

Examples

$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.
$TV = V \times V$.
$TS^1 = S^1 \times \mathbb{R}$ by $(p, v) \mapsto (p, a)$.

IMAGE 2

So any vector field v may be written as $v = a\frac{\partial}{\partial \theta}$
If there exist $v_1, \ldots, v_n$ such that $v_1(p), \ldots, v_n(p)$ is a basis, then $TM = M \times \mathbb{R}^n$.
$T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$.
Counterexample: $TS^2 \ncong S^2 \times \mathbb{R}^2$.

Example

Take an embedding $M \hookrightarrow \mathbb{R}^k$. Then $TM \hookrightarrow T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$ is also an embedding.

IMAGE 3

The map $(q, v) \mapsto q + v$ is not useful, but $(q, v) \mapsto v \in \mathbb{R}^k$ is.
We have a Gauss map $M \xrightarrow{G} \mathrm{Gr}(n, k)$ by $M \ni q \mapsto T_qM \in \mathrm{Gr}(n, k)$.

Example

M a hypersurface $M^n \subset \mathbb{R}^{n+1}$

IMAGE 4

$q \mapsto T_qM$ which is "=" to the normal line.

$$M \to \mathrm{GR}(n+1, n) \overset{\perp}{\leftrightarrow} \mathbb{RP}^n$$

$$p \longmapsto v_p$$

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & S^n \subset \mathbb{R}^{n+1} \arrow[d] \\ & \mathbb{RP}^n \end{tikzcd}$$

Definition: Riemannian Metric

A Riemannian metric is a smooth map $g : TM \to \mathbb{R}$ such that for each $p \in M$, $g|_{T_pM}$ is a positive-definite quadratic form where $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$.
In local coordinates, $(x_1, \ldots, x_n)$ and with $p \in M$ and $v \in T_p m$, $g_p(v) = \sum g_{ij}(p) v_i v_j$ where $g_{ij}(p)$ is symmetric.
We have that positive quadratic forms are in one-to-one correspondence with inner products.

$$\langle v, w \rangle = \frac{1}{2}(g(v + w) - g(v) - g(w))$$

In coordinates,

$$\langle v, w \rangle = \sum g_{ij}(p) v_i w$$

Example

Take an immersion $M \hookrightarrow \mathbb{R}^k$

IMAGE 5

The inner product in $\mathbb{R}^k$ restricts to a Riemannian metric on M.

Theorem

Every manifold admits a Riemannian metric.

Proof 1

Since any manifold may be immersed $M \hookrightarrow \mathbb{R}^k$, we may define a Riemannian metric.

Proof 2

Take a locally finite cover by coordinate charts, $\{(U_i, \phi_i) = (x_1, \ldots, x_n)\}$.
Within every chart, take a Riemannian metric g_i such that $\left\langle \frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_r} \right\rangle = \delta_{sr}$.
Take a partition of unity $f_i > 0$ subordinated to U_i $(\operatorname{supp} f_i \subset U_i$ and $\sum f_i \equiv 1)$. Then

$$g = \sum f_i g_i$$
$$g(v) = \sum f_i(p) g_i(v) > 0$$

with $v \neq 0$, $g_i > 0$ and $f_i > 0$ for some i.

Definition: Unit Tangent Bundle

Take some manifold M and a Riemannian metric g.
Then $g^{-1}(1) =: STM$ is the unit tangent bundle.

Claim 1

The unit tangent bundle is a smooth submanifold of $\operatorname{codim} = 1$ and $\dim = 2n - 1$.

Proof 1

Consider $\phi_\lambda : TM \to TM$ by $v \mapsto \lambda v$ with $\lambda \neq 0$.
Since $g(\phi_\lambda(v)) = \lambda^2 g(v)$, it follows that (exercise) a is a regular value of g if and only if $\lambda^2 a$ is a regular value. That is, for

$$M \overset{\phi}{\to} M \overset{f}{\to} \mathbb{R}$$

p is a regular value of $f \circ \phi$ if and only if $\phi(p)$ is a regular value of f and a is a regular value of f if and only if $\lambda^2 a$ is a regular value of $\lambda^2 f$.

Proof 2

(p, v) is a regular point if and only if there exists some $w \in T(TM)$ such that $L_w g \neq 0$.

IMAGE 6

For $w = v$,

$$\frac{\partial}{\partial \lambda} g(\lambda v)|_{\lambda=1} = \frac{\partial}{\partial \lambda} \lambda^2 g(v)|_{\lambda=1} = 2g(v) \neq 0$$

Claim 2

If g_0 and g_1 are Riemannian metrics on M, then $STM_{g_0} \cong STM_{g_1}$ are diffeomorphic.
For V, g_0 and g_1 with S_1 and S_2,

IMAGE 7

The map $x \mapsto \frac{g_0(x)}{g_1(x)} x$.

Exercise

If $M \hookrightarrow \mathbb{R}^k$ with an induced metric on M, then $STM \hookrightarrow \mathbb{R}^k \times S^{k-1}$ by $(p, v) \mapsto v \in S^{k-1}$ induces $STM \to S^{k-1}$.

IMAGE 8

Example

$$STS^2 \cong SO(3) \cong \mathbb{RP}^3$$

IMAGE 9

Take $p \perp v$, $||p|| = ||v|| = 1$, we can generate an orthonormal basis $(p, v, p \times v)$.
Recall that $S^n / x \sim -x = \mathbb{RP}^n$, but that we may also identify $B^n / x \sim -x$ on S^{n-1}.
That is, $\mathbb{RP}^3 = B^3 / \sim$ (with radius π).

IMAGE 10

Take $A \in SO(3)$.

November 7, 2024

Theorem: (Weak) Whitney Embedding Theorem

For M^n,

1. there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.
2. there exists and immersion $M \hookrightarrow \mathbb{R}^{2n}$.

Proof

(For simplicity, consider M compact)

$$\begin{tikzcd} M \arrow[r, hook, "F"] \arrow[dr, "F_v"'] & \mathbb{R}^m \ni v \arrow[d, "\pi_v"] \\ & \mathbb{R}^{m-1} \end{tikzcd}$$

Where $||v|| = 1$.

Hope: for m sufficiently large, F_v is an embedding and immersion.

What could go wrong?

- F_v not 1-1.

IMAGE 1

$p - q = t \cdot v$, $\frac{p-q}{||p-q||} = \pm v$.
So take $\Phi : M \times M \setminus \Delta \to S^{m-1}$ (where Δ is the diagonal) by $(p, q) \mapsto \frac{p-q}{||p-q||}$.
We need $v \notin \Phi(M \times M \setminus \Delta)$. Since $2n < m - 1$, this map has zero measure.

- F_v not an immersion.

IMAGE 2

Take $\psi : STM \to S^{m-1}$ by $(p, w) \mapsto w$. Then $v \notin \psi(STM)$.
Then F_v is an immersion where $2n - 1 < m - 1$ ($2n < m$).
Then for immersions, we may continue the process until $2n = m$; for embeddings, only until $2n + 1 = m$.

Remarks

$\Psi : TM \to \mathbb{R}^k$ by $(p, w) \mapsto w$.
$\Phi : (M \times M) \times \mathbb{R} \to \mathbb{R}^k$ by $(p, q, t) \mapsto (p - q)t$.

Homework

Problem 1

Take $L, M \subset \mathbb{R}^k$ submanifolds such that $\dim L + \dim M < k$.
Then for almost all x, $(x + L) \cap M = \varnothing$.

Problem 2

Take $M^n \subset \mathbb{R}^k$, $k > 2n$, and a projection $\rho_x : p \mapsto \frac{p-x}{||p-x||}$.

IMAGE 3

For almost all $x \in \mathbb{R}^n$, ρ_x is an immersion.

Problem 3

Take $L \subset M \times N \xrightarrow{\pi} N$.

IMAGE 4

Then $\pi^{-1}(q) \cap L = L_q$.
For almost all q, L_q is a smooth submanifold. Hint: $\pi|_L$.

R Smooth Topology

Take M compact and $0 \le r \le \infty$.
For $f : M \to \mathbb{R}^k$ (with either $k = 1$ or considered compontent-wise), define

$$||f||_{C^0} = \sup_{x \in M} |f(x)|$$

Then $C^0(M, \mathbb{R}^k)$ is a Banach space with induced distance $d(f,g) = ||f-g||_{C^0}$.
For $1 \le r < \infty$, take a finite coordinate cover (U_ℓ, ϕ_ℓ) (which is contained within some coordinate cover (W_ℓ, ϕ_ℓ). For $r = 1$, define

$$||f||_{C^1} = ||f||_{C^0} + \sum_{i,\ell} \left| \frac{\partial f}{\partial x_i^{(\ell)}} \right|$$

where $f = f \circ \phi_\ell$ and we may equivalently take $\max\{i, \ell\}$ rather than the sum.
Then $C^1(M, \mathbb{R})$ is a Banach space with well-defined topology. Similarly, for arbitrary r,

$$||f||_{C^r} = ||f||_{C^{r-1}} + \sum_{\ell, i_1, \dots, i_r} \left| \frac{\partial^r f}{\partial x_{i_1}} \cdots \partial x_{i_r} \right|$$

with equivalence again to a maximum. Finally, for $r = \infty$, $c^\infty \subset C^r$ for any finite r.
The C^∞ topology is generated by all C^r topologies.

$$C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^r \supset C^{r+1} \supset \cdots \supset C^\infty$$

Convergence: $f_i \xrightarrow{C^r} f$ if and only if $f_i \to f$, and so do all partial derivatives up to r $\frac{\partial f_i}{\partial x_c} \to \frac{\partial f}{\partial x_c}$.
Note that $C^\infty(M, \mathbb{R})$ is not a Banach space. It is, however, Fréchet.

N Smooth Topology

Instead, take $C^r(M, N)$ with $0 \le r \le \infty$ and M compact. Define

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right] & N \arrow[r, hook] & \mathbb{R}^k \end{tikzcd}$$

Theorem

For M compact, $r \geq 1$

1. the set of immersion $M \hookrightarrow \mathbb{R}^k$ is C^r open.

2. the same is true for embeddings.

Proof

Let $F : M \to \mathbb{R}^k$ be an immersion and G C^1 close to F.

$$\begin{tikzcd} T_p(M) \arrow[r, "DF"] \arrow[dr, "DG"] & T_{F(p)}\mathbb{R}^k & \text{rank} = n \\ & T_{G(P)}\mathbb{R}^k & \text{rank} = n \end{tikzcd}$$

Theorem

For $M^n \overset{C^r}{\to} \mathbb{R}^k$ with M compact and $r \geq 1$, if $k \geq 2n$ then immersions form a C^r open and dense set.
If $k \geq 2n+1$ then the same is true for embeddings.

Lowering the Dimension

Take $F : M \to \mathbb{R}^k$. We want an embedding/immersion approximation.
We know that there exists an embedding $M \overset{G}{\hookrightarrow} \mathbb{R}^M$ for sufficiently large m, so consider $\tilde{F} := (F, \varepsilon G) : M \to \mathbb{R}^k \times \mathbb{R}^m$.
Note that $(F, \varepsilon G) \overset{C^r}{\approx} (F, 0)$.
Take a basis of $\mathbb{R}^m$, $(e_1, \dots, e_m)$. We can take vectors close to our basis $v_1 \approx e_1$, $v_2 \approx e_2$, etc. We want to construct a series of projections $\mathbb{R}^k \times \mathbb{R}^m \overset{\pi_{v_1}}{\to} \mathbb{R}^k \times \mathbb{R}^{m-1} \overset{\pi_{v_2}}{\to} \cdots \to \mathbb{R}^k$.

IMAGE 5

Review: Ordinary Differential Equations

On $\mathbb{R}^n$,

$$x' = v(x) \text{ on } U \subset \mathbb{R}^n$$

with $x = (x_1, \dots, x_n)$. We want a map x from some open interval to U satisfying

$$\begin{cases} x_1'(t) = v_1(x(t)) \\ \vdots \\ x_n'(t) = v_n(x(t)) \end{cases}$$

with $x(0) = p$. Geometrically

IMAGE 6

were $x'(t) = v(x(t))$ is an integral curve.

Theorem

1. Existance: for every $p \in U$ and $t_0 \in \mathbb{R}$, there exists $\varepsilon > 0$ such that there is an integral curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ (where $\gamma(t_0) = p$.

2. Uniqueness: let $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ and $\eta : (t_0 - \delta, t_0 + \delta) \to U$ be two integral curves through p at t_0. Then $\gamma = \eta$ on the common domain.

We may refine (a) as

1. for every p and t_0, there exists $\varepsilon > 0$ on a neighborhood $V \ni p$ such that for every $q \in V$ there exists an integral curve $\gamma_q : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ where $\gamma_q(t_0) = q$.

2. $\gamma_q(t)$ is smooth in q and t. Namely, on $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$.

IMAGE 7

With Parameters

Consider $x' = v(x, \lambda)$. Then we have p and $\gamma_{p,\lambda}$ which depends smoothly on λ.

$$x = v(x, \lambda) \quad (p, \lambda) \quad \lambda' = 0$$

For $x' = v(x, t)$, given p and t_0 we have γ_{p,t_0} which depends smoothly on p and t_0.

$$x = v(x, t) \quad (p, t_0) \quad t' = 1$$

November 12, 2024

Recall: Solutions to ODEs

Given $x' = v(x)$ on U.
For each $p \in U$, there exists $\varepsilon > 0$ such that $\gamma_p : (t_0 - \varepsilon, t_{0+\varepsilon}) \to U$ where $\gamma_p(t_0) = p$.
Two solutions with the same initial conditions agree on their common domain.

Proof

Choose $p = 0$ and $t_0 = 0$, and write $\gamma'(t) = v(\gamma(t))$ with $\gamma(0) = 0$ for $t \in (-\varepsilon, \varepsilon)$. Then,

$$\gamma(t) = \int_0^t v(\gamma(\tau))\, d\tau$$

Take $B = C^0([-\varepsilon, \varepsilon], \mathbb{R}^n)$ with $\gamma(0) = 0$, and a map $\Phi : B \to B$ given by

$$\Phi(\gamma) = \int_0^t v(\gamma(\tau)\, d\tau$$

Claim: Φ is a contraction provided that ε is small. Then $\gamma = \Phi(\gamma)$ has a unique solution by the contraction mapping theorem.

Uniqueness

Fix t_0 and p. Then there exists a maximal integral curve. That is, given $\gamma_p : (a,b) \to \mathbb{R}^n$ and $\tilde{\gamma}_p : (a',b') \to \mathbb{R}^n$, $\gamma_p = \tilde{\gamma}_p$ on $(a',b') \cap (a,b)$ and they are defined on $(a,b) \cup (a',b')$.

Examples

On $\mathbb{R}^n$

- $v(p) = 0$ implies $\gamma_p(f) \equiv p$
- $v(x) = Ax$ gives $\gamma_p(t) = e^{At} p$.
- $v(x) = 1$ on $U = (0,1)$ with $\gamma_p(t) = p + t$

IMAGE 1

- $t_0 = 0$, $x' = \lambda x^2$, $\frac{dx}{dt} = \lambda x^2$, $\frac{dx}{\lambda x^2} = dt$, and $x = \gamma_p(t)$, gives

$$\int_p^x \frac{dy}{\lambda y^2} = \int_0^t dt$$

and

$$\frac{2}{\lambda y}\Big|_p^x = t$$

implies

$$\frac{-1}{\lambda x(t)} + \frac{1}{\lambda p} = t \quad \Longleftrightarrow \quad x(t) = \frac{p}{1 - \lambda t p}$$

IMAGE 2

So the max interval is $\left(-\infty, \frac{1}{\lambda p}\right)$.

Definition: Integral Curve

Let M be a manifold and v a vector field. $\gamma : (a,b) \to M$ is an integral curve if $\gamma'(t) = v(\gamma(t))$.

IMAGE 3

$\tau \to \gamma(t + \tau)$.

IMAGE 4

$(\phi \circ \gamma)' = (\phi_* v)(\phi(\gamma(t)))$.

Existence

For v a vector field on M, and for each $p \in M$ and t_0, there exists $\varepsilon > 0$ and an integral curve $\gamma_p : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ with $\gamma_p(t_0) = p$.

Definition: Complete Vector Field

v is complete if every maximal integral curve has $\mathbb{R}$ as a domain.

Examples

$v(x) = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete.
$v(x) = \frac{\partial}{\partial x}$ on $(0,1)$ is not complete.
$v(x) = Ax$ on $\mathbb{R}^n$ is complete.
If $||v(x)|| < a||x|| + b$ on $\mathbb{R}^n$, $v(x)$ will be complete.

Definition: Support of a Vector Field

$\operatorname{supp} v = \overline{\{x : v(x) \neq 0\}}$

Theorem

Suppose v is compactly supported. Then v is complete.

Corollary

If M is closed, then every vector field is complete.

Phase Portraits

Take $v = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

IMAGE 5

Consider $x' = -x$, then $x(t) = e^{-t}p$. Then it takes infinite time to travel from p to 0.

IMAGE 6

Definition: Flow

A flow is a 1-parameter subgroup in $\mathrm{Diff}(M)$, the group of diffeomorphisms on M. That is, $\phi^t \in \mathrm{Diff}(M)$, $t \in \mathbb{R}$ such that

- $\phi^0 = \mathrm{id}$
- $\phi^{t+\tau} = \phi^t \phi^\tau$

Note that this implies $(\phi^t)^{-1} = \phi^{-t}$.
Said differently, $\mathbb{R} \to \mathrm{Diff}(M)$ by $t \mapsto \phi^t$ is a group homomorphism.
Example: $\phi^t = e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is a one parameter subgroup $\mathrm{GL}(n) \subset \mathrm{Diff}(\mathbb{R}^n)$.
$F : M \times \mathbb{R} \to M$ given by $(x,t) \mapsto \phi^t(x)$ gives $F(x,0) = x$ and $F(F(x,t),\tau) = F(x, t+\tau)$.

Definition: Local Flow

Take $\phi^t(x)$ defined only on $t \in (-\varepsilon(x), \varepsilon(x))$.
For a neighborhood $U \subset M \times \{0\} \subset M \times \mathbb{R}$, $F : U \to M$.

IMAGE 7

Theorem

There is a 1-1 correspondence between local flows and vector fields; similarly, there is a correspondence between flows and complete vector fields.
Explicitly, $\phi^t \mapsto v(x) = \frac{d}{dt}\phi^t(x)|_{t=0}$ and, conversely, $v \mapsto \gamma'_p(t) =: \phi^t(p)$.

Proof

First, we need that $\phi^t : p \mapsto \gamma_p(t)$ is a flow.
Flow lines are intergral curves of the respective vector field.

$$\underbrace{\phi^{t+\tau}(p)}_{=\gamma_p(t+\tau)} = \underbrace{\phi^t(\phi^\tau(p))}_{=\gamma_{\gamma_p(\tau)}(t)}$$

So $t \mapsto \gamma_p(t+\tau)$ and $0 \mapsto \gamma_p(\tau)$ while $t \mapsto \gamma_{\gamma_p(\tau)}(t)$ while $0 \mapsto \gamma_p(\tau)$.

IMAGE 8

Therefore, by uniqueness, these are the same.
Now we need $v \rightsquigarrow \phi^t \rightsquigarrow w$ where $w = v$. Compute

$$w(p) = \frac{d}{dt}\underbrace{\phi^t(p)}_{\gamma_p(t)}\Big|_{t=0} = \frac{d}{dt}\gamma_p(t)\Big|_{t=0} = v(p)$$

where $\gamma'_p(t) = v(\gamma_p(t))$ and $\gamma_p(0) = p$.
Finally, we want $\phi \rightsquigarrow v \rightsquigarrow \psi$ where $\psi = \phi$. We have

$$\frac{d}{dt}\psi^t(p) = v(\psi^t(p)) \quad \text{and} \quad v(q) = \frac{d}{dh}\phi^h(q)\Big|_{h=0}$$

and we need $\frac{d}{dt}\phi^t(p) = v(\phi^t(p))$. Compute

$$\frac{d}{dt}\phi^t(p) = \frac{d}{dh}\psi^{t+h}(p)\Big|_{h=0} = \frac{d}{dh}\phi^h\underbrace{(\phi^t(p))}_{q}\Big|_{h=0} = v(\phi^t(p))$$

Examples

Take $\mathbb{R}^n$ and $v = v_0$ a constant.

IMAGE 9

Solutions have the form $\phi^t(p) = p + v_0 t$.
Take instead $v(x) = Ax$, $\phi^t(p) = e^{At}p$ and $e^{A(t+\tau)} = e^{At}e^{A\tau}$

Remark

In general, $e^{A+B} \neq e^A e^B$ but they are equal when $[A, B] = AB - BA = 0$.

November 14, 2024

Proposition

Let v be a vector field and $v(p) \neq 0$, then there exist $x_1, \dots, x_n$ near p such that $v(x) = \frac{\partial}{\partial x_n}$ and $\phi^t(x) = (x_1, \dots, x_{n-1}, x_n + t)$.

IMAGE 1

Proof

Take $y_1, \dots, y_{n-1}, y_n$, $p = 0$, and $v(p) = \frac{\partial}{\partial y_n}$.

IMAGE 2

Then there exists time τ such that $y_n(\phi^{-\tau}(q)) = 0$ where $\tau(q)$ depends on q.
Define a map $\pi : q \mapsto \phi^{-\tau}(q)$. Then $q = \phi^{\tau(q)}(\pi(q))$. Then

$$\begin{aligned} x_1 &= y_1 \circ \pi \\ &\vdots \\ x_{n-1} &= y_{n-1} \circ \pi \\ x_n &= \tau(x) \end{aligned}$$

Lie Brackets Via Flows

Take two vector fields v, w and consider the flow ϕ^t of v

IMAGE 3

Then we may take $D\phi^t W_{\phi^{-t}(p)} \in T_p M$ and define $-\frac{d}{dt} D\phi^t W_{\phi^{-t}(p)}\Big|_{t=0} = L_v w(p)$, the Lie derivative.
Compare with

IMAGE 4

where $L_v f(p) = -\frac{d}{dt} f(\phi^{-t}(p))\Big|_{t=0}$ which depends only on $v(p)$.

Remark

If M is compact, then ϕ^t is defined for all times.
Consider $\mathfrak{X}(M)$ of C^∞ vector spaces which is topologically Fréchet (complete, locally convex, with an invariant metric). Then

$$D\phi^{-t}(w) = w_t \in \mathfrak{X}(M) \quad \text{and} \quad L_v w = -\frac{d}{dt} w_t\Big|_{t=0}$$

Example 1

Take v a constant vector flow on $\mathbb{R}^n$ with $\phi^t(x) = x + tv$ and $D\phi^t = \mathrm{id}$. Then for $w = \sum w_i \frac{\partial}{\partial x_i}$,

$$L_v w = -\frac{\partial}{\partial t} \sum w_i(x - tv) \frac{\partial}{\partial x_i}\Big|_{t=0} = \sum L_w w_i(x) \frac{\partial}{\partial x_i}$$

Example 2

Let $w = Ax$ with v constant. Then

$$L_v w = -\frac{d}{dt}A(x - vt)\Big|_{t=0} = Av$$

Example 3

Let $v = Ax$ with w constant. Then $\phi^t(x)e^{At}x$ and $D\phi^t = e^{At}$ and

$$L_v w = -\frac{d}{dt}e^{At}w\Big|_{t=0} = -Aw$$

Note that examples 2 and 3 demonstrate the skew symmetry of the Lie bracket.

Example 4

Let $v = Ax$ and $w = Bx$. Then $\phi^t(x) = e^{At}x$, $D\phi^t = e^{At}$, and

$$L_v w = -\frac{d}{dt}\underbrace{e^{At}}_{D\phi^t}\underbrace{Be^{-At}}_{w_{\phi^{-t}(x)}}\Big|_{t=0} = -(AB - BA) = -[A, B]$$

Theorem

$L_v w = [v, w]$.

Proof

Let $K = \{p \,:\, v(p) = 0\}$ and $U = M \setminus K$.
Then $M = U \coprod \partial K \coprod \mathrm{int}(K)$. Since U and the interior of K are together dense, and the flow for $\mathrm{int}(K)$ is $\phi^t = \mathrm{id}$, we need only examine U.
Then with the local normal form near p, write

$$w = \sum w_i(x)\frac{\partial}{\partial x_i}, \qquad L_v w = \sum \frac{\partial w_i}{\partial x_n}\frac{\partial}{\partial x_i} \quad \text{and} \quad [v, w] = \sum \frac{\partial w_i}{\partial x_n}\frac{\partial}{\partial x_n}$$

Remark

Consider flows $v \rightsquigarrow \phi^t$ and $w \rightsquigarrow \psi^t$.
The bracket $[v, w]$ measures to what extent ϕ^t and ψ^s do not commute.

Example

$[A, B]$ measures to what extent e^{At} and e^{Bs} do not commute.

Theorem

$[v, w] = 0$ if and only if $\phi^t\psi^s = \psi^s\phi^t$.

Observation 1

Write $D\phi^t = \phi^t_*$ and observe that $\phi^t_* v = v$ while $\psi^s_* w = w$.

IMAGE 5

Proof

Consider $F : M \to M$. Recall that $v \mapsto F_* v$ by $[\eta] \mapsto [F \circ \eta]$. Compute

$$\eta(\tau) = \gamma_p(\tau) = \phi^\tau(p)$$

so $\frac{d}{dt}\phi^t(p) = v$ and $\phi^t \eta(\tau) = \phi^t(\phi^\tau(p))\phi^{t+\tau}(p)$. Therefore

$$\frac{d}{dt}\phi^{t+\tau}(p) = \frac{d}{dt}\phi^\tau(\phi^t(p)) = v(\phi^t(p))$$

Observation 2

Next, observe that for $F : M \to M$, $F\phi^t = \phi^t F$ if and only if $F_* v = v$.

Proof

Recall that $v_p = [t \mapsto \phi^t(p)]$.
If $F\phi^t = \phi^t F$, then $F\phi^t(p) = \phi^t F(p)$.

IMAGE 6

If instead, $F_* v = v$, then $F \circ \phi^t \circ F^{-1} = \phi^t$. Left as an exercise.

Proof

($\Longleftarrow$) Set $\psi^s =: F$ such that, by previous observation, $\psi^s_* v = v \in \mathfrak{X}(M)$. Write

$$-[w, v] = -L_w v = \frac{d}{ds}\psi^s_* v = 0$$

($\Longrightarrow$) If $[v, w] = 0$, then $\psi^s_* v = v$ for $F := \psi^s$. By the second observation, $\psi^s \phi^t = \phi^t \psi^s$. To see that ψ^s_* is constant, compute

$$\frac{d}{ds}\psi^s_* v = \frac{d}{dh}\psi^{s+h}_* v\Big|_{h=0} = \frac{d}{dh}\psi^s_*\psi^h_* v\Big|_{h=0} = \psi^s_* \frac{d}{dh}\psi^h_* v\Big|_{h=0} = -\psi^s_*[w, v] = 0$$

Theorem

IMAGE 7

If $c(t) = \psi^{-t}\phi^{-t}\psi^t\phi^t(p)$, then $c'(0) = 0$ and $c''(0) = 2[v, w](p)$.

Fact

If $c : (-\varepsilon, \varepsilon) \to M$, $c(0) = p$, and $c'(0) = 0$, then $c''(0) \in T_p M$ is well-defined.

November 19, 2024

Recall

The vector fields v and w give rise, respecitvely, to flows ϕ^t and ψ^t.

IMAGE 1

Theorem

$c(0) = 0$, $c'(0) = 0$ and $c''(0) = 2[v, w](p)$.

Example

Take $v(x) = Ax$ ad $w(x) = Bx$ such that $\phi^t(x) = e^{At}x$ and $\psi^t = e^{Bt}x$.
Then $c(t) = e^{-Bt}e^{-At}e^{Bt}e^{At}x$. Recall that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots \quad \text{and} \quad e^{Bt} = I + Bt + \frac{1}{2}B^2t^2 + \cdots$$

So

$$\begin{aligned} c(t) &= (I - Bt + \frac{1}{2}B^2t^2 + \cdots)(I - At + \frac{1}{2}A^2t^2 + \cdots)(I + Bt + \frac{1}{2}B^2t^2 + \cdots)(I + At + \frac{1}{2}A^2t^2 + \cdots)x \\ &= [I + (-B - A + B + A)t + (B^2 + A^2 + BA - B^2 - BA - AB - A^2 + BA)t^2]x \\ &= [I + 0 + [B, A]t^2 + \cdots]x \end{aligned}$$

We have $v(x) = v_0 + Ax + \cdots$ and $w(x) = w_0 + Bx + \cdots$. If $v = v_0 + Ax + o(x)$ $\phi^t(x) = e^{At}x + A^{-1}e^{At} - I)v_0 + \cdots$ where the higher order terms are given by some $\text{const}(x)t^3$.

Theorem

If $v(0) \neq 0$, then in some coordinate system $v = \frac{\partial}{\partial x_n} = v_0$.
Therefore $\phi^t(x) = x + tv_0$.

Definition: Distribution

A distribution E of rank k on M^n is a field of k-dimension $E_p \subset T_pM$ depending smoothly on points on M.
Note: $\text{Gr}_k(T_pM) \ni E_p$.

Depending Smoothly

- pick a chart

IMAGE 2

Then $T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$, $\phi_*(E_p) \in \text{Gr}_k(\mathbb{R}^n)$ and we have a smooth map $V \to \text{Gr}_k(\mathbb{R}^n)$ by $p \mapsto \phi_*(E_p)$.

- local coordinates

  There exist $v_1, \ldots, v_k$ such that $v_1(x), \ldots, v_k(x)$ are linearly independent and $\text{span}(v_1(x), \ldots, v_k(x)) = E_x$.

Examples

- Let $v \neq 0$ be a non-vanishing vector field. Then $v \cdot \mathbb{R} = \operatorname{span}(v(p)) =: E_p$.
- Take $\mathbb{R}^n \supset \mathbb{R}^k$ with $E_p = \operatorname{span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)$

IMAGE 3

- $E = \operatorname{span}\left(\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) \subset \mathbb{R}^3$.

Definition: Integral Manifold

An integral manifold of E is a 1-1 immersion $N \hookrightarrow M$ such that $T_pN = E_p$.

Examples

- For $v \neq 0$, the integral manifolds are flow lines.
- For $\mathbb{R}^n \supset \mathbb{R}^k$ with $E_p = \operatorname{span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)$, the integral manifolds are $\mathbb{R}^k + p$ (horizontal lines).
- Take $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

IMAGE 4

With $v = \frac{\partial}{\partial x} + a\frac{\partial}{\partial y}$ with a irrational which are dense in $\mathbb{T}^2$ and this not embedded.

- $E = \operatorname{span}\left(\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) \subset \mathbb{R}^3$ has no integral manifold.

  Consider $\mathbb{R}^3 \supset \mathbb{R}^2$ with $\operatorname{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \operatorname{span}(v, w)$. Then $L_v f = \frac{\partial f}{\partial x} = 0$ and $L_w f = \frac{\partial f}{\partial y} = 0$. So if $L_v f = 0 = L_w f$, then f is identically constant. For our example, we have $[v, w] = \frac{\partial}{\partial y} \notin E$.

Definition:

E is said to be integrable (or involutive) if for any v, w tangent to E, $[v, w]$ is also tangent to E.

Theorem: Frobenius

The following are equivalent

1. E is integrable.
2. locally E looks like $\mathbb{R}^k \subset \mathbb{R}^n$.
3. through every point, there exists an integral manifld.

   That 2 implies 3 can be seen from the diagram; that 3 implies 1 comes from $E_p = T_pN$.

Proof 1 Implies 2

Let us consider the two dimensional case. Take $E = \operatorname{span}(v, w)$.

IMAGE 5

Notice that $E' = \operatorname{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$. We have that $E \ni [v, w] \neq 0$ if and only if $[v, w] = fv + gw$.
Then for $u = av + bw$, $\operatorname{span}(v, u) = \operatorname{span}(v, w)$ and $[v, u] = 0$. Therefore

$$
\begin{aligned}
[v, av + bw] &= 0 \\
(L_v a) \cdot v + L_v b \cdot w + b \cdot [v, w] &= 0 \\
L_v a \cdot v + L_v b \cdot w + bfv + bgw &= 0
\end{aligned}
$$

and

$$
\begin{cases}
L_v a + bf = 0 \\
L_v b + bg = 0
\end{cases}
$$

Then, setting $h := \ln b$ such that $b = e^h$ $(b \neq 0)$, $L_v \ln b = -g$. Consider $L_v h = -g$

IMAGE 6

So $v = \frac{\partial}{\partial y}$, $h(x, y) = h(x) - \int_0^y g(x, t)\, dt$ and $L_v h = \frac{\partial h}{\partial y} = -g(x, y)$. We may repeat this argument on $L_v a = -bf$.
Then $E = \operatorname{span}(v, u)$ where $[v, u] = 0$ which implies their flows commute $\phi^t \psi^s = \psi^s \phi^t$.
Consider the map $B^2 \ni (t, s) \to \phi^t \psi^s(p)$ which is a local $\mathbb{R}^2$-action.

IMAGE 7

So $x(q) = t$ and $y(q) = s$.
This proof may be generalized to arbitrary dimension.

Example: Foliation

Consider $S^3 \subset \mathbb{C}^2$. $S^1 = U(1) = e^{i\theta}$ acts on C^2 by $e^{i\theta} \cdot z = (e^{i\theta} z_1, e^{i\theta} z_2)$. This admits no foliation.
Consider instead two solid torii.

IMAGE 8

$S^3 \to \mathbb{CP}^1 = S^2$.

December 3, 2024

Definition: Transversality

Let $K \xrightarrow{F} M$ for $M \supset N$ with K compact and N a proper submanifold.
$F \pitchfork N$ (f is transverse to N) if $\forall x \in K$ such that $F(x) \in N$, $DF_x(T_x k) + T_{F(X)} N = T_{F(x)} M$.

Example 1

Take $K \overset{F}{\hookrightarrow} M$ a submanifold. For all $x \in K \cap N$, $T_x K + T_x N = T_x M$.

Example 2

Assume $\dim K + \dim N < \dim M$. Then $K \pitchfork N$ if and only if $K \cap N = \varnothing$.

Example 3

Take $F : K \to M$ and $N = \{q\}$. Then $F \pitchfork q$ if and only if q is a regular value.

Example 4

Take $M = N \times B$ and $N = N \times b$.

IMAGE 1

Then $F \pitchfork N \times b$ if and only if b is a regular value of $\pi \circ F$.

Proposition

Assume $F \pitchfork N$. Then $F^{-1}(N) \subset K$ is a submanifold.

$$\dim F^{-1}(N) = \dim K + \dim N - \dim M$$

Proof (Ideas)

Take $X \subset K$ and assume there exists a cover U_i of X such that $X \cap U_i$ is a submanifold in U_i. Then X is a submanifold. Locally, we can streamline.

IMAGE 2

Locally, the linear subspace n can be thought of as a projection and it looks like Example 4 above.

Theorem: Transversality

$\{F \pitchfork N \ : \ F \in C^r(K, M)\}$ for $1 \leq r \leq \infty$ is open and dense for $r < \infty$ and residual if $r = \infty$ (a countable intersection of open and dense sets).

Proposition

Take $G : K \times X \to M \pitchfork N$. Then for almost all $x \in X$, $G_x : K \times x \to M \pitchfork N$.

IMAGE 3

Where $G_x \pitchfork N$ for almost all x.

Proof

$G^{-1}(N) \subset K \times X \xrightarrow{\pi} X$.
Claim: $x \in X$ is a regular value of $\pi|_{G^{-1}(N)}$ if and only if $G_x \pitchfork N$.
Then $D\pi : T_{(p,x)}L \twoheadrightarrow T_xX$ (regular value). Adding the vertical direction, $T_{(p,x)}L + T_{(p,x)}(K \times x) = T_{(x,p)}(K \times X)$ (transversality).

IMAGE 4

Applying $DG_{(p,x)}$, since $T_{(p,x)}L$ is the kernel of DG, $DG_X(T_{(x,p)}(K \times x)) + T_{G(x,p)}N = T_{G(x,p)}M$.
Transversality holds when $K \xrightarrow{f} \mathbb{R}^n = M \supset N$.
Start with F. We need $\tilde{F} \pitchfork N$ arbitrarily close to F.
So take X to be a small, closed ball in $\mathbb{R}^n$. Write $G(p,x) = F(p) + x$. Then $G \pitchfork N + X$ and consequently $G \pitchfork N$. Therefore $G_x = F + x \pitchfork N$ for almost all x.

Theorem: Tubular Neighborhood

Let $M^m \subset \mathbb{R}^n$ closed. Then there exists an open set U and a submersion $\rho : U \to M^m$ with $\rho^{-1}(y)$ a disk of dimension $n - m$.

IMAGE 5

(U, ρ) is a tubular neighborhood.

Proof

Write $N = \{(y,v) \; : \; y \in M, \; v \in T_y\mathbb{R}^n \perp T_yM\} \subset M \times \mathbb{R}^n$, a smooth submanifold. This is the normal bundle.
Take V to be a neighborhood of M in the normal bundle (i.e. $||v|| < \varepsilon$). Then there exists a map $\Phi : V \to \mathbb{R}^n$ given by $(y,v) \mapsto y + v$.
Claim: Φ is a diffeomorphism on its image (for ε small). Therefore $\rho(\Phi(y,v)) = y$.
To prove the claim, we need that Φ is a local diffeomorphism and one-to-one.
Write $D\Phi_{(y,0)} : T_{(y,0)}N \to T_y\mathbb{R}^n$. Therefore $T_{(y,0)}N = T_yM \oplus N_y \to T_yM \oplus N_y$, and $D\Phi_{(y,v)}$ remains invertible when $||v|| < \varepsilon$.
Now take $x_k = (y_k, v_k)$ and $x_k' = (y_k', v_k')$ such that $\Phi(x_k) = \Phi(x_k')$. If no valid ε exists, then we would have $||v_k|| \to 0$ and $||v_k'|| \to 0$. But then $x_k \to x$ and $x_k' \to x'$ for $x, x' \in M$.
Φ restricted to M is inclusion, and in the limit $\phi(x_k) = \phi(x_k')$. However, the map is a local diffeomorphism which means it is locally injective, so we have a contradiction.

Proof of Proposition

Take $N \subset M \subset \mathbb{R}^n$ and $K \xrightarrow{F} M$. We need $\tilde{F} : K \to M \pitchfork N$ and X a closed ball in $\mathbb{R}^n$. Using the tubular neighborhood projection, define $G := \rho(F(p) + x)$ such that $G : X \times K \to M$.

IMAGE 6

Then we have transversality of the neighborhood of $F(K) \subset M$ which implies $G_x = \rho(F + x) \pitchfork N$ for almost all x.

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Theorem

Consider $\{F : K \overset{C^r}{\to} M \supset N\}$ for K compact with boundary. This is open and dense for $r < \infty$ and residual for $r = \infty$. Then $F \pitchfork N$ and $F|_{\partial K} \pitchfork N$.

Proposition

$F \pitchfork N$ and $F|_{\partial K} \pitchfork N$ implies that $F^{-1}(N)$ is a smooth submanifold with boundary $\partial K \supset \partial F^{-1}(N) = (F|_{\partial K})^{-1}(N)$.

Theorem Jet (Thom's) Transversality Theorem

Fix a Riemannian metric $\langle \cdot, \cdot \rangle$. Then for f, $Df : T_p m \ni v \mapsto L_v f \in \mathbb{R}$.
Since $Df = \left(\frac{\partial f}{\partial x_i}, \ldots, \frac{\partial f}{\partial x_n}\right)$, $\langle \nabla f, \cdot \rangle = Df$. So $\nabla f \in \mathcal{X}(M)$ and $\nabla f : M \to TM$.

IMAGE 1

Functions which satisfy $\nabla f \pitchfork M$ are called Morse Functions.
Note that we cannot apply our standard version of the Transversality Theorem since, in general, we cannot find f to satisfy $v = \nabla \tilde{f}$.

IMAGE 2

Definition: Degree Mod 2

Let M^n and N^n closed and $F : M \overset{C_1}{\to} N \ni y$ a regular value.
We define $\deg F = \#\{F^{-1}(y)\} \pmod 2$ (Note: given information regarding the orientation, this is not modulo 2).

Theorem

The degree of F is well defined independent of y.
The degree is a smooth-homotopy invariant.

Examples

1. $\deg \mathrm{id} = 1$
2. $F : S^1 \ni z \mapsto z^k \in S^1$, $\deg F = k$.
3. Polynomials $P : S^2 \to S^2$, $\deg P = k$.
4. For any k, there exists $F : S^n \to S^n$ such that $\deg F = k$.

IMAGE 3

This is the suspension of the n-sphere. We take the map $F(\theta, t) = (\tilde{F}(\theta), t)$

1. Take $M_1 \overset{F_1}{\to} N_1$ and $M_2 \overset{F_2}{\to} N_2$ to construct $(F_1, F_2) : M_1 \times M_2 \to N_1 \times N_2$. Then $\deg(F_1, F_2) = \deg F_1 \cdot \deg F_2$.

2. $A : T^n \to T^n$ for $A \in M_n(\mathbb{Z})$, $\deg A = \det A$.

3. Consider $F : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$ for $|x| >> R$.

IMAGE 4

Proof

Let y_0, y_1 be regular values connected by a path L. Make F transverse to L. That is, $\tilde{F} \pitchfork L$.
Then F is a local diffeomorphism at $F^{-1}(y_0)$ and $F^{-1}(y_1)$. Under small perturbation, so is $\tilde{F}$. So $\#F^{-1}(y_0) = \#\tilde{F}^{-1}(y_0)$ and $\#F^{-1}(y_1) = \#\tilde{F}(y_1)$.

IMAGE 5

So $\tilde{F}^{-1}(L)$ is a one dimensional compact submanifold of M with boundary, and $\partial\tilde{F}^{-1}(L) = \tilde{F}(y_0) \cup \tilde{F}(y_1)$. This one dimensional submanifold is either a circle, an interval or a collection of such elements. In any case, the number of boundary points is even so $\#\tilde{F}(y_0)$ and $\#\tilde{F}(y_1)$ sum to an even parity and therefore share parity.

Smooth Homotopy

$F_0 \sim F_1$ are smoothly homotopic if there exists a map $g : M \times [0,1] \to N$such that $g|_{M\times 0} = F_0$ and $g|_{M\times 1} = F_1$.

IMAGE 6

Take y a regular value for G, F_0, F_1 and consider $G^{-1}(y)$.

IMAGE 7

In this case, the previous parity arguments follow similarly.

Smooth Approximation of Continuous Functions

Take $F : M \overset{C^0}{\to} N$ and $\tilde{F} : M \to N$. Then

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[rr, bend left, "G"] & N \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

IMAGE 8

$U \overset{\rho}{\to} N$ and $\tilde{F} = \rho \circ G$.

Proposition

$\deg F := \deg \tilde{F}$ is well-defined.
Claim: any two smooth approximations are smoothly homotopic.
We need (a family) $\tilde{F}_t$ such that $\tilde{F}_0 \overset{\tilde{F}_t}{\sim} \tilde{F}_1$.

$$\begin{tikzcd} M \arrow[r, "\tilde{F}_0"] \arrow[r, bend right, "\tilde{F}_1"'] \arrow[rr, bend left, "G"] & N \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

Write $G_t = (1-t)\tilde{F}_0 + t\tilde{F}_1$. Then $G_t \to U \subset \mathbb{R}^n \overset{\rho}{\to} N$, so $\rho \circ G_t = \tilde{F}_t$.

IMAGE 9

Theorem: Brower Fixed Point

For $F : \overline{B^n} \overset{C_0}{\to} \overline{B^n}$, F has a fixed point. That is, ther eexists x such that $F(x) = x$.

Proof

Assume not, that $F(x) \neq x$.

IMAGE 10

We know that $G : \overline{B^n} \overset{C^0}{\to} S^{n-1}$ and $G|_{S^{n-1}} = \mathrm{id}$ which has $\deg = 1$.
Lemma: For $M = \partial X$ compact, if this map commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & N \\ X \arrow[ur, "G"] & \end{tikzcd}$$

then $\deg f = 0$.

IMAGE 11

But from above, if $f = G|_{S^{n-1}}$ we have a a contradiction $1 \neq 0$.

Fundamental Theorem of Algebra

Write $P(z) = z^k + a_1 z^{k-1} + \cdots + a_k$ for $k > 1$, a polynomial $P : S^2 \to S^2$, with $\deg P = k \geq 1$.
Assume for sake of contradiction that $P(z) = 0$ is not a regular value, but then 0 is not in the image and $\deg F = 0 \neq k$

Euler Characteristic Mod 2

Take M closed, $v : M \to TM$ a vector field such that $v \pitchfork M$.

IMAGE 12

F is a vector field if and only if $\pi \circ F = \mathrm{id}$. The transversality theorem does not necessarily preserve this features.

IMAGE 13

$\phi(x) = \pi F(x)$. We claim that $\phi : M \to M$ is a diffeomorphism. Then define $\tilde{v} = F \circ \phi^{-1}$.
Then define $\chi_2(M) = \#(v(M) \cap M)$.

Theorem

χ_2 is well defined.

Remarks

If $v_0, v_1 \pitchfork M$, $w_t = (1-t)v_0 + tv_1$, $w : M \times [0,1] \to TM$ are supersets of $w^{-1}(M) \to M$.

Example

Take S^2 and the vector field generating rotation about the z axis where $\chi_2 = 0$.

IMAGE 14

This vector field is invariant under antipodal involution. So $S^2/\sim = \mathbb{RP}^2$, but $\chi_2 = 1$. So these are clearly not diffeomorphic.