Topics in Analysis (F24)

September 30, 2024

Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over $\mathbb{C}$) is a normed vector space with algebraic operations

$$\begin{aligned} (x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication} \end{aligned}$$

and a norm

$$x \mapsto ||x||$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $||xy|| \leq ||x|| \cdot ||y||$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad ||e|| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded inear operators $A : X \to X$) equipped with algebraic operations

$$
\begin{aligned}
(A+B)x &= Ax+Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}
$$

and the operator norm

$$||A||_{\mathcal{L}(X)} = \sup_{x\neq 0} \frac{||Ax||_X}{||x||_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.
The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n\times n}$.

$$A = (a_{ij})_{i,j=1}^n \qquad Ax = y \qquad \sum_{j=1}^n a_{ij}x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in $\mathbb{C}^n$ leadsto a norm in $\mathbb{C}^{n\times n}$

$$
\begin{aligned}
||(x_i)|| &= \left(\sum |x_i|^2\right)^{1/2} & ||A|| &= \\
||(x_i)|| &= \sum |x_i| & ||A|| &= \max_j \sum_i |a_{ij}| \\
||(x_i)|| &= \max |x_i| & ||A|| &= \max_i \sum_j |a_{ij}|
\end{aligned}
$$

All norms are quivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \to \mathbb{C}$ continuous and $||f|| = \max_{t\in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\operatorname{int}(K) \neq 0$, $f : K \to \mathbb{C}$ continuous where f is holomorphic on $\operatorname{int}(K)$ and

$$||f|| = \max_{t\in K} |f(t)| = \max_{t\in K\setminus \operatorname{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^{\infty}(\mathbb{N})$ or $B = L^{\infty}(S, \sigma, \mu)$ with (S, σ, μ) a measure space, $f : S \to \mathbb{C}$ essentially bounded functions and

$$||f|| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N)}} \left(\sup_{t \in S \setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum |x_n|$ and $||f|| = \int_{\mathbb{R}^d} |f(t)| \, dt$ respectively.
Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t) g(t) \, dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on $\mathbb{R}^d$ with bounded variation.
Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^{n} |\mu(A_i)| < +\infty.$$

Then, $f \, dm = d\mu$ gives $L^1(\mathbb{R}^d) \to M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n \times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \to \mathbb{C}^{n \times n}$ and norm

$$||f||_B = \max_{t \in k} ||f(t)||_{C^{n \times n}}.$$

Then $B \cong (C(K))^{n \times n}$ the $n \times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2) \end{aligned}$$

and norm

$$||(b,\lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0,1)$. One writes $(b,\lambda) = (b,0) + \lambda(0,1) = b + \lambda \cdot e$.
In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b,0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B-algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0,\ \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \varnothing$ be a subset of a Banach algebra B.
The Banach subalgebra generated by M is the smallest closed subalgebra containing M.

$$\operatorname{alg} M = (\operatorname{clos} \operatorname{alg}_B M)$$

- Remark

  $\operatorname{alg} M$ is the intersection of all closed subalgebras containing M.
  $\operatorname{alg} M = \operatorname{clos}\left\{\sum_{i=1}^{N} \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)}\right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Exammple 1

Take B unital, $b \in B$. Then

$$\operatorname{alg}\{e, b\} = \operatorname{clos}_B \left\{ \sum_{i=0}^{N} \lambda_i b^i \ :\ \lambda_i \in \mathbb{C},\ N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \to B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

  $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$

- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.
Then $\phi^{-1} : B_2 \to B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

October 2, 2024

Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \text{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \to L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x.

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2}z = (x_1 + x_2)z = x_1z + x_2z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1x_2}z = (x_1x_2)z = x_1(x_2z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that ϕ is a homomorphism.
To show that ϕ is an isometry,

$$||L_x|| = \sup_{z\neq 0} \frac{||L_xz||}{||z||} \geq \frac{||L_xe||}{||e||} = \frac{||x||}{1} = ||x||.$$

Then also ϕ is injective and $\mathrm{im}\,\phi$ is closed. Since $\mathrm{im}\,\phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.
Right-regular representation:

$$R_x = z \mapsto zx$$

Section 1.2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.
Note that $y = x^{-1}$ is uniquely determined.
Write GB for the set of all invertible elements of B.

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $||x|| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \cdots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \to S$,

$$(e-x)s_n = s_n(e-x)e - x^{n+1}.$$

Taking $n \to \infty$

$$(e-x)s = s(e-x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.
Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x+y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty}(-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty}(-x^{-1}y)^n x^{-1}$$

so

$$||(x+y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1}||y||^n = \frac{||x^{-1}||^2||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \to 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdroff, $f : K \to \mathbb{C}$ continuous.
$GB = \{f \in C(K) \, : \, f(t) \neq 0, \ \forall \, t \in K\}$.

Example 2

$B = C^{n \times n}$.
$GB = \{A \in \mathbb{C}^{n \times n} \, : \, \det A \neq 0\}$.

2.5 Definition:

Let $G_0 B$ stand for the connected componet of GB containing e.

Remarks

- the ε-neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B \, : \, ||x - y|| < \varepsilon\}$$

  For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

$$\sigma : [0,1] \ni \lambda \mapsto y_1 \lambda + y_2(1 - \lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).

- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB.

- GB is the union of disjoint (path-)components where each component is both open and closed in GB.

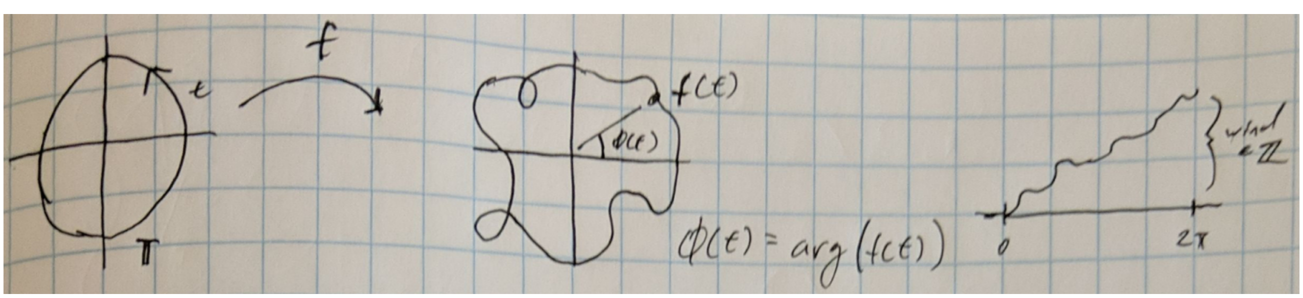
- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0,1] \to GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.

- $G_0 B = \{x \in GB \, : \, \exists$ a path in GB connecting e and $x\}$.

2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} \, : \, |z| = 1\}$ and continuous functions $f : \mathbb{T} \to \mathbb{C}$.
GB is the non-vanishing continuous functions $f : \mathbb{T} \to \mathbb{C}$ ($f(t) \neq 0, \ \forall t \in \mathbb{T}$).
For $f \in GB$ one can define a winding number.



We have $\frac{1}{2\pi} \arg f(e^{ix})$ a continuous function with

$$\text{wind}(t) = \left[\frac{1}{2\pi} \arg f(e^{ix})\right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\mathrm{wind}(t) \in \mathbb{Z}$.
The map $GB \ni f \mapsto \mathrm{wind}(t) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).
Therefore $G_0 C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \mathrm{wind}(f) = 0\}$. In fact, we will see that we have equality.
That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

$G_0 B$ is a normal subgroup of GB.

Proof

- $G_0 B$ is a group.

  For any $x, y \in G_0 B$, there exist paths $\gamma_1 : [0,1] \to GB$ and $\gamma_2 : [0,1] \to GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.
  Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0 B$.
  Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in GB$.

- $G_0 B$ is a normal subgroup of GB.

  For every $y \in GB$, $yG_0By^{-1} \subseteq G_0 B$ if and only if $yG_0B = G_0By$.
  Take $x \in G_0 B$ with path γ, then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1) yxy^{-1} \in G_0 B.$$

2.8 Definition: Abstract Index Group

The quotient group GB/G_0B is called the abstract index group of B.

Remark

GB/G_0B is in 1-to-1 correspondence with the set of connected components of GB.
Indeed, the (path-)connected components of GB are given by $yG_0B = G_0By$ (for $y \in GB$).

$$y_1 G_0 B = y_2 G_0 B \iff y_2^{-1} y_1 G_0 B = G_0 B \iff y_2^{-1} y_1 \in G_0 B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.
For $xy = yx$, we have $\exp(x+y) = \exp(x)\exp(y)$.
In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{n=0}^{k} \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges.
Continuity left as an exercise. Need to show:

$$\left|\left|\sum \frac{x^n}{n!} - \sum \frac{y^n}{n!}\right|\right| \leq ||x-y|| \cdot M_{x,y}$$

The fact that $\exp(x+y) = \exp(x)\exp(y)$ follows from multiplying terms and the binomial formula.

October 7, 2024

Recall

$GB\ e+x$.
G_0B connected component of GB containing e.
GB/G_0B is the abstract index group.
$B = C(\mathbb{T}) \rightsquigarrow f \in GC(\mathbb{T}) \rightsquigarrow \text{ind}(f)$.

Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

Lemma:

For $y \in B$, $||y|| < 1$, there exists $x \in B$ such that $\exp(x) = e + y$.

Proof

Define

$$\log(e+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots \in B.$$

This converges absolutely ($||y|| < 1$), therefore it converges in B by completeness.

Identities

$$\exp(\log(e+y)) = \sum_{n=0}^{\infty} \frac{\left(\sum_k \frac{y^k}{k}(-1)^{k-1}\right)^n}{n!} = e + y$$

Proof

G_0B is equal to the set of all finite products of exponentials of elements in B.

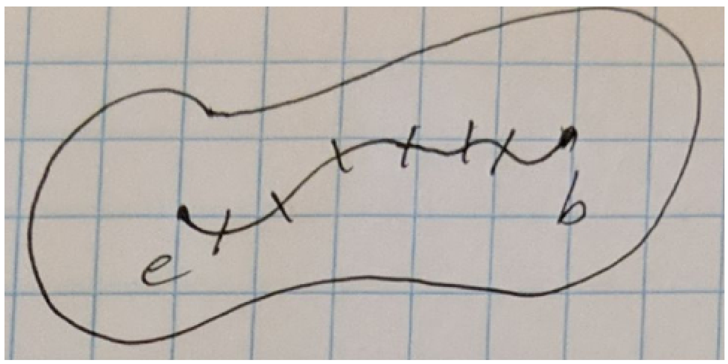
$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{\exp(a_1)\exp(a_2)\cdots\exp_{a_n} \in B\}$$

Proof

Call $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$.
Then observe that each Γ_n is path-connected and contains e.
For $b = \exp(a_1)\cdots\exp(a_n) \in \Gamma_n$, define a path

- $\sigma : [0,1] \to \Gamma_n$
- $\sigma(t) = \exp(ta_1)\cdots\exp(ta_n)$ is continuous with $\sigma(0) = e$ and $\sigma(1) = b$.

Therefore, Γ is path-connected and contains e. It follows that $\Gamma \subseteq G_0B$.
To prove that $G_0B \subseteq \Gamma$, take $b \in G_0B$ and show that there exists a path in GB $\gamma : [0,1] \to GB$ continuous with $\gamma(0) = e$ and $\gamma(1) = b$.



We have that $(\gamma(t))^{-1}$ is continuous and bounded in the norm. Then $\gamma(t)$ is uniformly continuous.
$||\gamma^{-1}(t)|| \leq M$.
$(\exists N) : |t-s| \leq \frac{1}{N} \implies ||\gamma(t) - \gamma(b)|| \leq \frac{1}{M} \cdot \frac{1}{2}$. Write

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1)\gamma^{-1}\left(\frac{N-1}{N}\right)\gamma\left(\frac{N-1}{N}\right)\gamma^{-1}\left(\frac{N-2}{2}\right)\cdots\gamma\left(\frac{1}{N}\right)\gamma^{-1}\left(\frac{1}{N}\right)\gamma(0) = \prod_{k=1}^{N}\gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right).$$

Therefore, with $s_k = \gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right)$, $b = \prod_{k=1}^{N} \exp(\log(s_k))$.

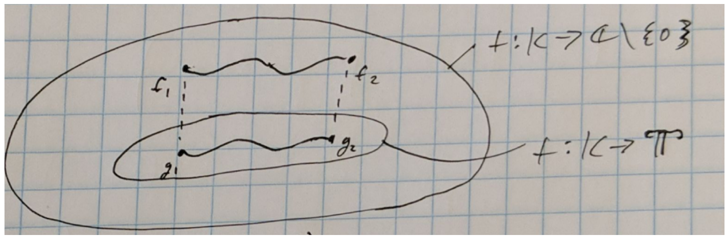
$$||s_k - e|| \leq ||\gamma^{-1}\left(\frac{k}{N}\right)|| \cdot ||\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)|| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

Corollary

If B is commutative, $G_0B = \{\exp(a) \; : \; a \in B\}$.

Remark

Special case: $B = C(K)$ (K compact Hausdorff space).
$G_0B = \{\exp(a) \; : \; a \in C(K)\}$.
GB/G_0B is an equivalence class of functions $f : K \to \mathbb{C} \setminus \{0\}$ with respect to path-connectedness.
That is, $f_1 \sim f_2$ if and only if there exists continuous $F(t,x) : [0,1] \times K \to \mathbb{C} \setminus \{0\}$ with $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$.
These are the homotopy classes of continuous functions $f : K \to \mathbb{C} \setminus \{0\}$.
This corresponds to homotopy classes of continuous functions $f : K \to \mathbb{T}$ (with $\mathbb{T} = \{z \in \mathbb{C} \; : \; |z| = 1\}$) called the 1st co-homotopy group of K $\pi^1(K)$.



$f : K \to \mathbb{C} \setminus \{0\}$ and $\frac{f}{|f|} : K \to \mathbb{C} \setminus \{0\}$ are path-connected by $\sigma(s) = \frac{f}{|f|^s}$, $s \in [0,1]$.
$f_1 \sim f_2$ in $K \to \mathbb{C} \setminus \{0\}$ implies that $\frac{f_1}{||f_1||} \sim \frac{f_2}{||f_2||}$ in $K \to \mathbb{T}$ by $F(s,x)$ and $\frac{F(s,x)}{|F(s,x)|}$.
We conclude that $\pi^1(K) \cong GC(K)/G_0C(K)$.

Example

Let $B = C(\mathbb{T})$.

$$G_0B = \{\exp(a) : a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) : \mathrm{wind}(f) = 0\}$$

For $f \in GC(\mathbb{T})$, $\mathrm{wind}(f) = 0$ implies that $f = \exp(a)$ has a logarithm.
This implies that $f \in G_0B$ which itself implies that $\mathrm{wind}(f) = 0$, since $\mathrm{wind}(f)$ is continuous on $GC(\mathbb{T})$ and therefore constant on the component.
Therefore, $GB/G_0B \cong \mathbb{Z}$ via the winding number.
For connected components of GB, define $\chi_n(t) = t^n$, $|t| = 1$, where $\mathrm{wind}(\chi_n) = n$.

Remark: Closed Subalgebras and Invertibility

Let A be a closed subalgebra of B (both being unital, $e \in A$, $e \in B$).
Obviously, if $a \in A$ is invertible in A (i.e. $a^{-1} \in A$) then a is invertible in B. Then $GA \subseteq GB \cap A \subseteq GB$.

Example

Take $B = C(\mathbb{T})$ and $A = \{f \in C(\mathbb{T}) : f_n = 0, \forall n < 0\} = C_+(\mathbb{T})$ where $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx$ is the nth Fourier coefficient.
Formally: $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$ in $B = C(\mathbb{T})$, $|t| = 1$.
$f \in A : f(t) = \sum_{n=0}^{\infty} f_n t^n$, $|t| = 1$ has an analytic extension into the unit disk $|t| < 1$.
More precisely, $\phi : A(\overline{\mathbb{D}}) \to C_+(\mathbb{T}) \subseteq C(\mathbb{T})$ by $f \mapsto f|_{\mathbb{T}}$.
Where $A(\overline{\mathbb{D}}) = \{f \in \overline{D} \to \mathbb{C}$ continuous, holomorphic on $\mathbb{D}\}$ and $\mathbb{D} = \{t \in \mathbb{C} : |t| \le 1\}$.
Then, for $f \in A(\overline{\mathbb{D}})$ with $n \in \{-1,-2,-3,\dots\}$,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\,dz = \lim_{r \to 1^-} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}}\,dz = 0$$

- In fact, ϕ is an isometry.

$$||f||_{A(\overline{\mathbb{D}})} = \sup_{|z| \le 1} |f(z)| = \max_{|z|=1} |f(z)| = ||f|_{\mathbb{T}}||_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since ϕ is not constant.

- ϕ is linear and multiplicative.

- $C_+(\mathbb{T})$ is a closed subset of $C(\mathbb{T})$.

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

  is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivaially, ϕ is surjective and $C_+(\mathbb{T})$ is an algebra.

Example

$\chi_1(t) = t$ is invertible in $C(\mathbb{T}) = B$.
$x_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$ while $\chi_1(t) \in C_+(\mathbb{T})$.
Therefore $GA \subseteq GB \cap A$ may not be equal.

Definition: Boundary

The boundary of a subset U of a topological space X is $\partial U = \overline{U} \setminus \operatorname{int}(U)$.

Remark

For $U \subseteq X$, $X = \operatorname{int}(U) \cup \partial U \cup \operatorname{int}(X \setminus U)$ a union of disjoint sets.

Lemma:

1. if $a \in \partial GA$, then $a \notin GA$ and there exists a sequence $a_n \in GA$ such that $a_n \to a$.

2. if $a \in \partial a$ and $a_n \in GA$ such that $a_n \to a$, then $||a_n^{-1}|| \to +\infty$.

Proof of 1

$a \in GA$ would imply $a \in \operatorname{int}(GA)$ and not a boundary point.

Proof of 2

Otherwise, there would exist a bounded subsequence $||a_{n_i}^{-1}|| \leq M$.

$$||a_{n_i}^{-1} - a_{n_j}^{-1}|| \leq ||a_{n_i}^{-1}|| \cdot ||a_{n_j} - a_{n_i}|| \cdot ||a_{n_j}^{-1}|| \leq M^2 ||a_{n_i} - a_{n_j}||$$

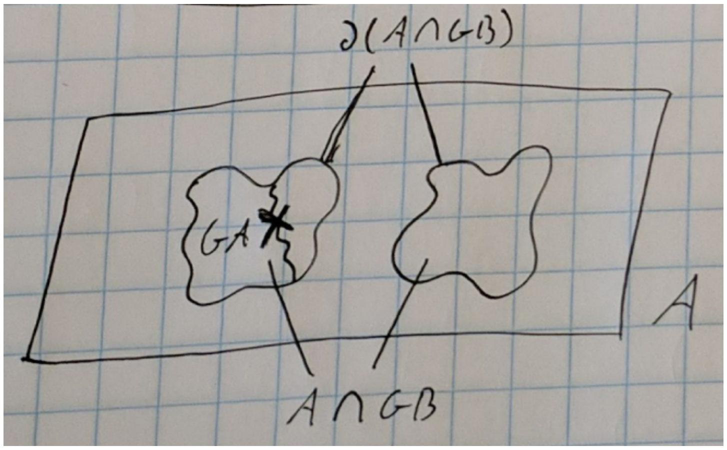
Since a_n converges, $\{a_n\}$ is Cauchy which implies $a_{n_i}^{-1}$ is Cauchy.
Then $a_{n_i}^{-1} \to b \in A$. $e = a_{n_i} a_{n_i}^{-1} \to ab$ implies $a^{-1} = b$ and $a \in GA$. However $a \notin GA$.

Proposition

Let A be a closed subalgebra of B ($e \in A$, $e \in B$). Then $\partial GA \subseteq \partial(A \cap GB)$ (both boundaries are considered in A).

Remark

Both GA and $A \cap GB$ are open subsets of A.



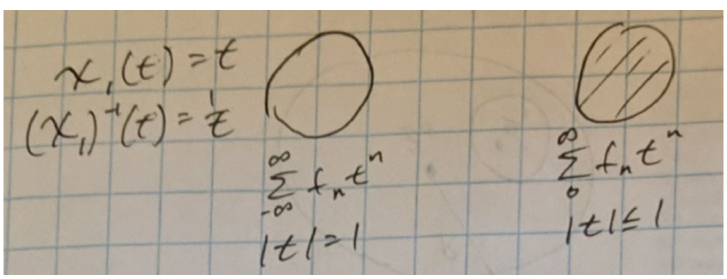
Proof

Take $a \in \partial GA$ and suppose $a \notin \partial(A \cap GB)$.
Take $a \in \partial GA$: $a_n \in GA$, $a \notin GA$, $a_n \to a$, $||a_n^{-1}|| \to +\infty$.

October 9, 2024

Recall

$A \subseteq B$, $GA \subseteq A \cap GB$.
If $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$ and $B = C(\mathbb{T})$.



Recall: Theorem

For GA, $A \cap GB$ open sets in A, $U \subseteq X$, $\partial U = \overline{U} \setminus \operatorname{int} U$, we have that $\partial GA \subseteq \partial(A \cap GB)$.

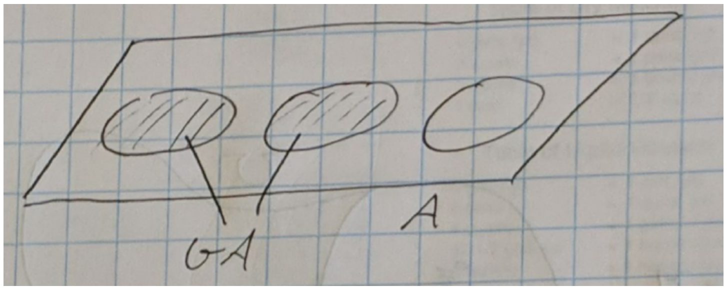
Proof

Take $a \in \partial GA$, $a_n \to a$, $a \notin GA$, $a \in A$.
Since $a_n \in GA$, $||a_n^{-1}|| \to +\infty$.
However, $a \notin GB$ otherwise $a \in GB$, $a_n \to a$ implies $a_n^{-1} \to a^{-1}$ (in GB) and, consequently, $\sup ||a_n^{-1}|| < +\infty$, a contradiction.
Therefore $a \notin A \cap GB$ and, consequently, $a \in \partial(A \cap GB) = \overline{(A \cap GB} \setminus (A \cap GB)$.

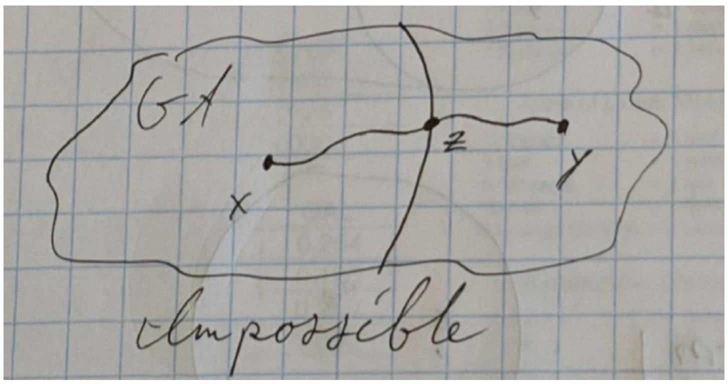
Theorem

Let A be a closed subalgebra of B.
GA is equal to the union of some components of $A \cap GB$.

Proof



Let U be a component of $A \cap GB$.
We want to show that either $U \cap GA \neq \varnothing$ or $U \subseteq GA$.



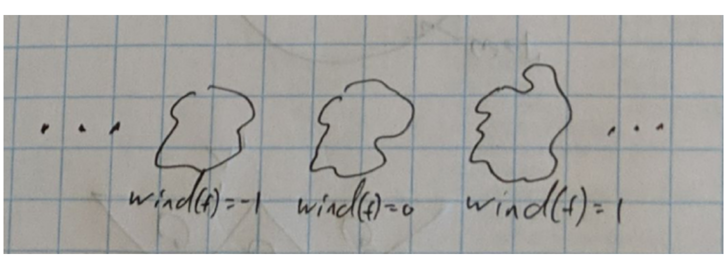
The above cannot occur since, by path-connectedness, for $x, y \in U,\ x \in GA,\ y \neq\in GA$, there would need to be some $z \in \partial GA$ with $z \notin A \cap GB$ a contradiction.
Alternatively, take $A \cap GB$ open in A.
Then $A \cap GB \cap \partial(A \cap GB) = \varnothing$ and $(A \cap GB) \cap \partial GA = \varnothing$ by the previous theorem.
Write $A = GA \cup \partial GA \cup \operatorname{int}(A \setminus GA)$. Then

$$A \cap GB = GA \cup \varnothing \cup \operatorname{int}(A \setminus GA) \cap (A \cap GB)$$

and $U = (GA \cap U) \cup \operatorname{int}(A \setminus GB) \cap U$ where $(GA \cap U) \cap \operatorname{int}(A \setminus GA) = \varnothing$ and open in U.
Therefore either $GA \cap U = \varnothing$ or $GA \cap U = U$ which implies that $U \subseteq GA$.

Example

Take $B(\mathbb{T})$ and $A = C_+(\mathbb{T}) \cong A(\overline{D})$.
Then $GB = \{f : \mathbb{T} \to \mathbb{C}\ :\ f(t) \neq 0$.



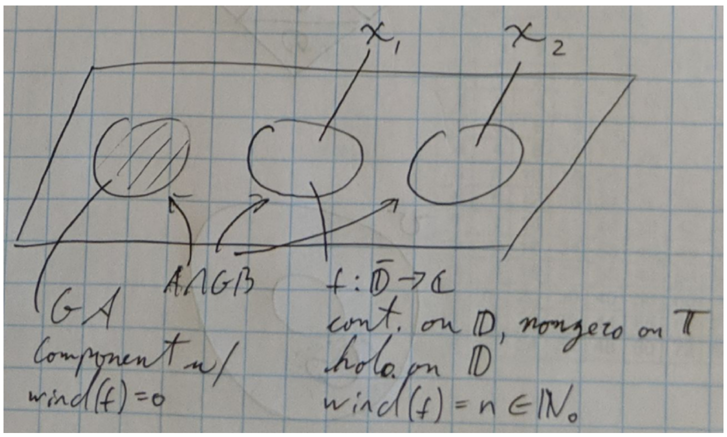
Then take

$$A \cap GB = \{f : \mathbb{T} \to \mathbb{C} \text{ continuous},\ f(t) \neq 0,\ |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that $f \in A \cap GB$ which implies $\mathrm{wind}(f) \in \{0,1,2,3,\cdots\}$ gives the number of zeroes of f inside $\mathbb{D}$.

$$\begin{aligned}
\mathrm{wind}(f) &= \frac{1}{2\pi i}\left[\log f(e^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r\to 1^-}\left[\log f(re^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r\to 1}\int_0^{\pi}\frac{f'(re^{ix})}{f(re^{ix})}ire^{ix}\,dx \\
&\underset{z=re^{ix}}{=} \frac{1}{2\pi i}\lim_{r\to 0}\int_{|z|=r}\frac{f'(z)}{f(z)}\,dz
\end{aligned}$$

Which gives the number of zeros of $f(z)$ inside $|z| < 1$



Section 1.3: Holomorphic Vector-Valued Functions

Goal

Define the notion of holomorphic/analytic functions $f : \Omega \to X$ where $\Omega \subset \mathbb{C}$ open and X a (complex) Banach space.

Sumary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

Theorem

For a function $f : \Omega \to X$, $\Omega \subseteq \mathbb{C}$ open and X Banach, the following are equivalent

1. f is differentiable at every $z_0 \in \Omega$, i.e. there exists $f'(z_0) \in X$ such that

$$\lim_{z\to z_0}\left|\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right|\right|_X = 0$$

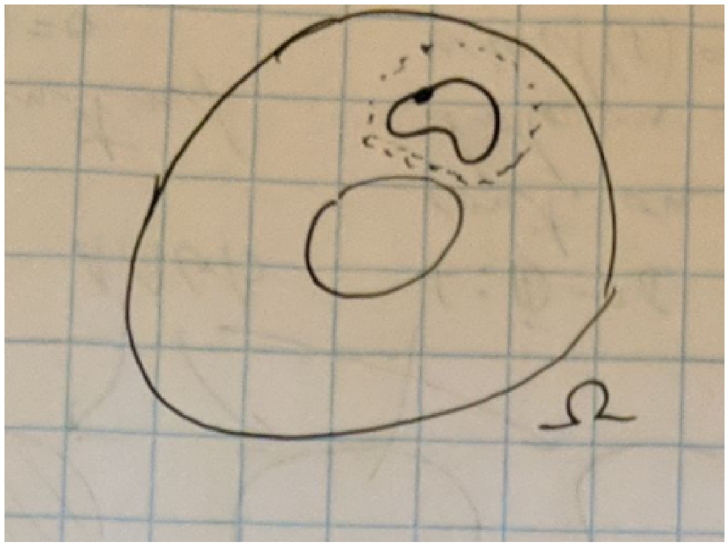
2. f is analytic at each point $z_0 \in \Omega$, i.e. f has a convergent power series at z_0 with radius of convergence $R_{z_0} > 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,\ |z-z_0| < R_{z_0},\ a_n \in X$$

which converges in the norm of X.

3. $f : \Omega \to X$ is continuous (in the norm) and for every piecewise smooth closed contour Γ contained in a disk D ($\Gamma \subseteq D \subseteq \Omega$).

$$\int_\Gamma f(z)\, dz = 0$$



Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then f is (strongly)-holomorphic.

Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour Γ can be parameterized by $\sigma : [0,1] \to \Omega$.

$$\int_\Gamma f(z)\, dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t) \text{ continuous}}\, dt$$

This is independent of the choice of parameterization.
Now $I = \int_0^1 h(t)\, dt$ can be defined via Riemann sums. Given a partition P, $h : [0,1] \to X$ continuous.

$$\lim_{\text{mesh}(P)\to 0} ||S(h,P,\xi) - I||_X = 0$$

where $S(h,P,\xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$, $P = \{x_0, x_1, \dots, x_n\}$, $\xi_i \in [x_{i-1}, x_i]$.
Note that h is uniformly continuous and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\text{mesh}(P_1) < \delta$, $\text{mesh}(P_2) < \delta$ implies

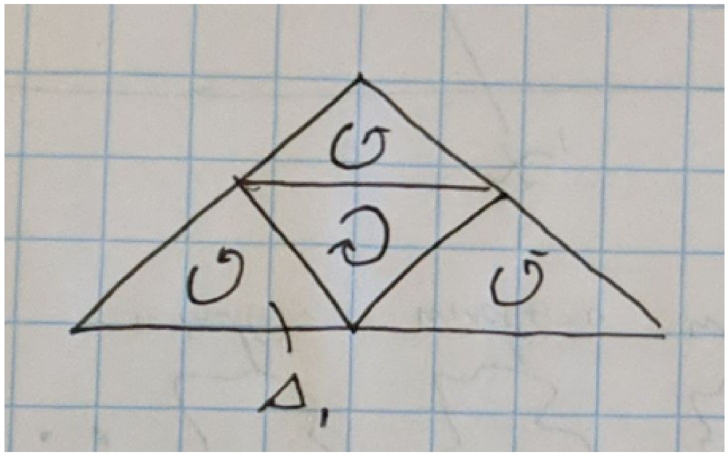
$$\left|\left|S(f,P_1,\xi^{(1)}) - S(f,P_2,\xi^{(2)})\right|\right| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $||\int_\Gamma f(z)\, dz|| \le \int_\Gamma ||f(z)||\, |dz| \le (\text{length}(\Gamma)) \sup_{z\in\Gamma} ||f(z)||$.

Sketch of Proof (1) to (3)

To show: $\int_\Delta f(z)\,dz = x_0 = 0$ by contradiction that $x_0 \neq 0$.

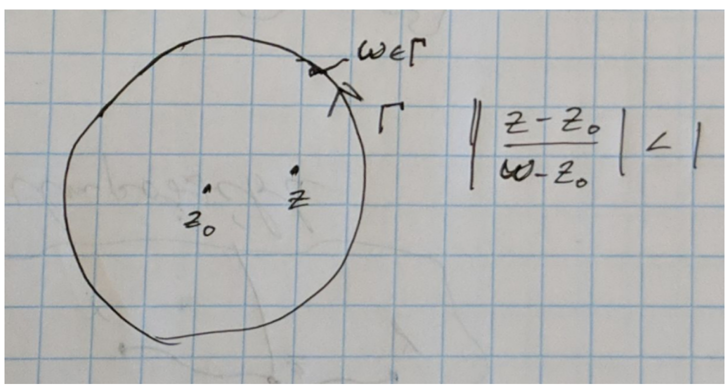


We have $\left|\left|\int_{\Delta_1} f\,dz\right|\right| \geq \frac{||x_0||}{4}$, $\left|\left|\int_{\Delta_n} f\,dz\right|\right| \geq \frac{||x_0||}{4^n}$.

Sketch of Proof (3) to (2)

$\int_\Gamma f\,dz = 0$ implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - z}\,d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{w - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{w - t}\right)^n$$

Therefore

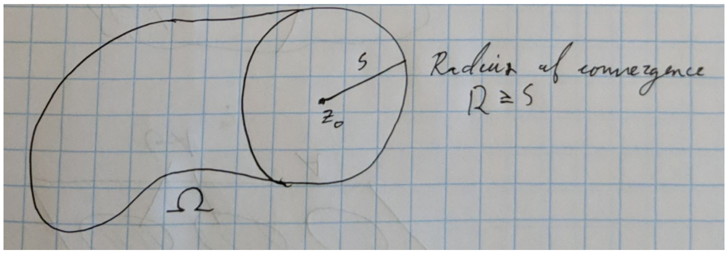
$$f(z) = \frac{1}{2\pi i}\int_\Gamma f(\omega)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\omega - z)^{n\pi}}\,d\omega = \sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{(\omega - z)^{n\pi}}\,d\omega = \sum_{n=0}^{\infty}(z-z_0)^n a_n$$

with the sequence converging (in X) on $|z - z_0| < |\omega - z_0|$.

- Radius of Convergence

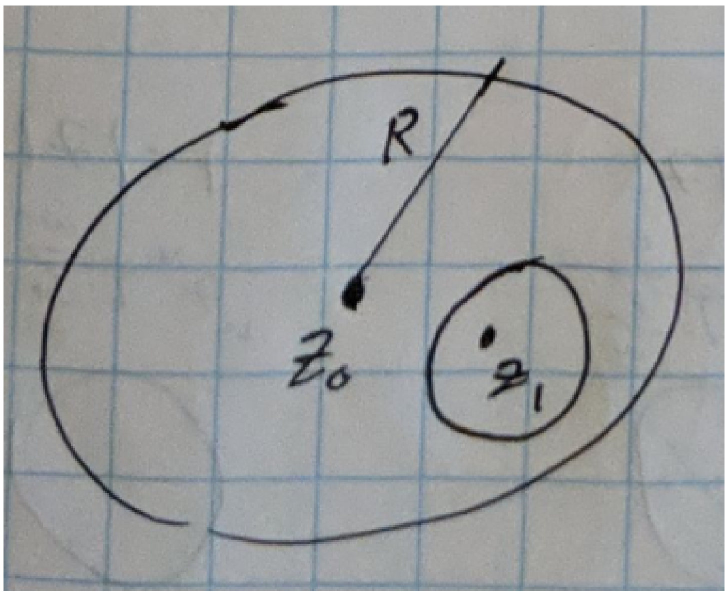
$$R^{-1} = \limsup_{n\to\infty} ||a_n||^{1/n}$$

(Root Test: $|z-z_0| < R$ convergence; $|z-z_0| > R$ divergence)



Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable, $f(z) = \sum a_n(z-z_0)^n$, then $f'(z) = \sum a_n \cdot n(z-z_0)^{n-1}$.
The radius of convergence is the same. This also implies that f is infinitely differentiable.



Take $z - z_0 = (z-z_1) + (z_1 - z_0)$ and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z-z_1)^k \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0) \right)$$

which converges for at least $|z-z_1| < R - |z_1 - z_0|$.

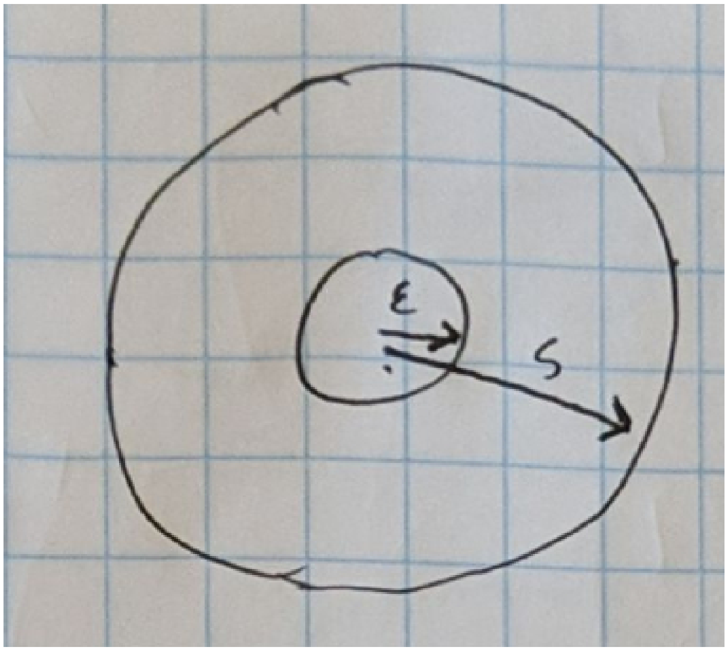
October 14, 2024

Theorem

Let $f : D_\varepsilon(z_0) \to X$ ($D_\varepsilon(z_0) = \{z \in \mathbb{C} \,:\, |z-z_0| < \varepsilon\}$) be holomorphic.
Then $R = S$ where

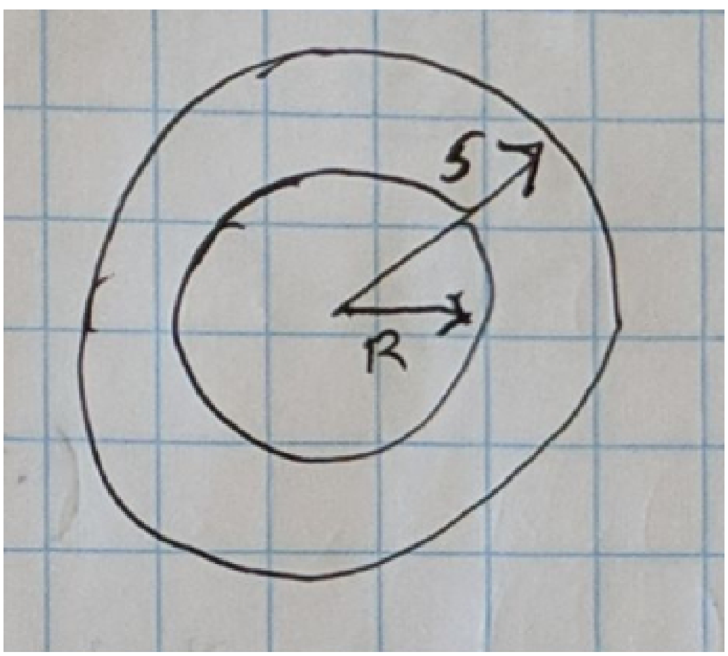
1. R is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ ($R^{-1} = \limsup_{n\to\infty} ||a_n||^{\frac{1}{n}}$).

2. S is the radius of the largest open disk $D_S(z_0)$ such that there exists an analytic extension of f from $D_\varepsilon(z_0)$ to $D_S(z_0)$.



Proof

By definition, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < R$. Then $|z-z_0| < R$ if and only if $\limsup_{n\to\infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$ if and only if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges. Therefore, it converges to a holomorphic function on $R \leq S$.
If $f(z)$ has an analytic extension to $D_S(z_0)$, see step (3) $\implies$ (2) of previous theorem.



Then $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega \cdot z}\, d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < r < S$ with $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{(z-\omega)^{n+1}}\, d\omega$.
From this, we conclude $R \geq S$.

Definition: (Weakly) Holomorphic Function

A function $f : \Omega \to X$ ($\Omega \subseteq \mathbb{C}$ open, X Banach) is called weakly holomorphic if $\phi \circ f : \Omega \to \mathbb{C}$ is holomorphic, $\forall \phi \in X^* = \mathcal{L}(X;\mathbb{C})$ bounded linear functionals.
A function $f : \Omega \to \mathcal{L}(X,Y)$ (X, Y Banach) is weakly-operator holomorphic if $h_{\phi,X} : \Omega \to \mathbb{C}$ is holomorphic for all $\phi \in Y^*$, $x \in X$ where $h_{\phi,X}(z) = \phi(f(z)x)$.

Remarks

Obviously: f strongly holomorphic $\implies$ f weakly holomorphic.

$$\left|\left|\frac{\phi(f(z+h))-\phi(f(z))}{h}-\phi(f'(z))\right|\right| \leq ||\phi||\cdot\left|\left|\frac{f(z+h)-f(z)}{h}-f'(z)\right|\right|$$

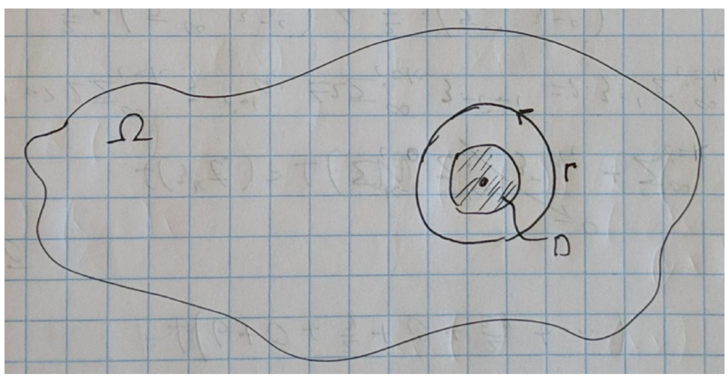
For $f:\Omega\to\mathcal{L}(X,Y)$: f strongly holomorphic $\implies$ f weakly holomorphic $\implies$ f weakly operator holomorphic.
For $x\in X$, $\phi\in Y^*$, $\Lambda_{x,\phi}:\mathcal{L}(X,y)\ni A\mapsto\phi(Ax)\in\mathbb{C}$ and $\Lambda_{x,\phi}\in(\mathcal{L}(X,y))^*$.
All the converses are also true.

Theorem (Dunford)

Take X Banach, $\Omega\subseteq\mathbb{C}$ open.
If $f:\Omega\to X$ is weakly holomorphic, then it is strongly holomorphic.

Proof

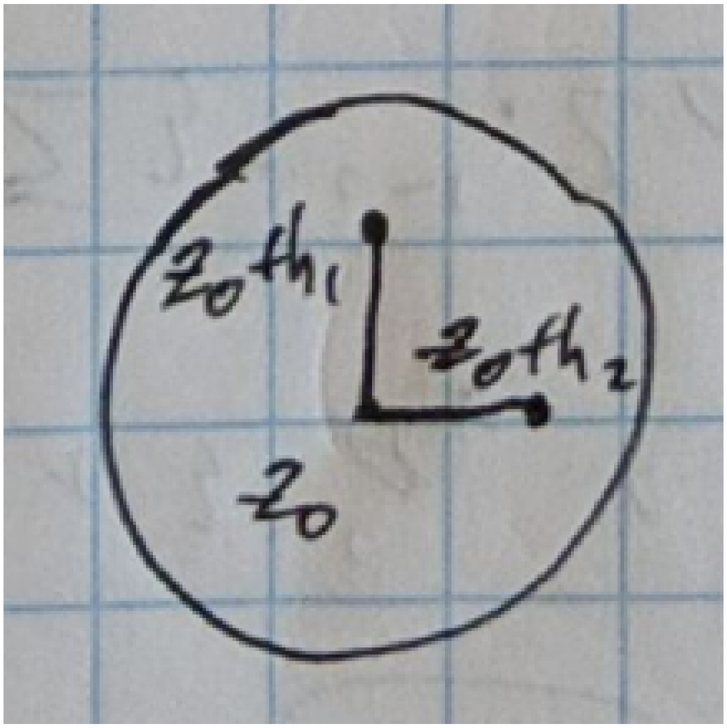
We want to show that for any $z_0\in\Omega$, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists in X.
Choose $\varepsilon>0$ such that the disk $D_\varepsilon(z_0)$ and circle $C_{2\varepsilon}(z_0)=\Gamma$ are in Ω.



For $\phi\in X^*$, $\phi(f(z))$ is holomorphic in Ω.

$$\phi(f(z))=\frac{1}{2\pi i}\int_\Gamma\frac{\phi(f(\omega))}{z-\omega}\,d\omega,\ z\in D$$

Apply this to $z=z_0$, $z=z_0+h_1$ and $z=z_0+h_2$ with $0<|h_1|<\varepsilon$, $0<|h_2|<\varepsilon$, $h_1\neq h_2$.



$$
\begin{aligned}
A_{h_1,h_2} &= \frac{1}{h_1-h_2}\left\{\frac{f(z_0+h_1)-f(z_0)}{h_1} - \frac{f(z_{0+h_2)-f(\omega)}}{h_2}\right\} \\
\phi(A_{h_1,h_2}) &= \frac{1}{h_1-h_2}\left\{\frac{\phi(f(z_0+h_1)-\phi(f(z_0))}{h_1} - \frac{\phi(f(z_0+h_2)-\phi(f(\omega))}{h_2}\right\} \\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{h_1-h_2}\left\{\frac{1}{h_1}\left(\frac{1}{z_0+h_1-\omega}-\frac{1}{z_0-\omega}\right)-\frac{1}{h_2}\left(\frac{1}{z_0+h_2-\omega}-\frac{1}{z_0-\omega}\right)\right\}d\omega \\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{h_1-h_2}\left\{\frac{1}{(z+h_1-\omega)(z_0-\omega}-\frac{1}{(z+h_2-\omega)(z_0-\omega)}\right\}d\omega \\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{(z_0+h_1-\omega)(z_0+h_2-\omega)(z_0-\omega)}\,d\omega
\end{aligned}
$$

Observe that the denominator is at least ε^3, therefore $|\phi(A_{h_1,h_2})| \le \frac{\varepsilon^3}{2\pi}\sup_{\omega\in\Gamma}||f(\omega)||\cdot||\phi||$ (so long as f continuous, which will be proven).
Therefore $\forall\phi\in X^*$,

$$\sup_{\substack{0<|h_1|<\varepsilon \\ 0<|h_2|<\varepsilon \\ h_{1\neq h_2}}} |\phi(A_{h_1,h_2})| < +\infty.$$

By the uniform boundedness principle, identify $A_{h_1,h_2}\in X$ with $X^{**} = \mathcal{L}(X^*,\mathbb{C})$.
Then $\sup_{h_1,h_2}||A_{h_1,h_2} < +\infty$ and

$$\left|\left|\frac{f(z_0+h_1)-f(z_0)}{h_1} - \frac{f(z_0-f(z)}{h_2}\right|\right| \le C\cdot|h_1-h_2|.$$

Now, for any sequnce $\{h_n\}_{n=3}^\infty$, $0<|h_n|<\varepsilon$, $h_n\to 0$,

$$\frac{f(z_0+h_n)-f(z_0)}{h_n}$$

is a cauchy sequence. Therefore $\lim_{n\to\infty} \frac{f(z_{0+h_n} - f(z_0)}{h_n}$ exists in X independent of choice of $\{h_n\}$. That is

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists in X.

Section 1.4: Spectrum and Resolvent

Consider a unital Banach algebra B.

Definition: Spectrum

For $b \in B$, the spectrum of b in B $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b$ is not invertible in $B\}$.

Definition: Resolvent

The resolvent is a function $R(b;\lambda) = (\lambda e - b)^{-1}$. $R(b,\cdot) : \mathbb{C} \setminus \sigma_B(b) \to B$.
$\mathbb{C} \setminus \sigma_B(b)$ is the resolvent set.

Theorem

1. The spectrum $\sigma_B(b)$ is a non-empty, compact subset of $\mathbb{C}$.

2. The resolvent $R(b,\lambda)$ is an analytic, Banach valued function on $\mathbb{C} \setminus \sigma_B(b)$.

Proof of (a)

$\sigma_B(b)$ is bounded, because $\lambda e - b$ is invertible for $|\lambda| > ||b||$.

$$\lambda e - b = \lambda\left(e - \frac{1}{\lambda} b\right)$$

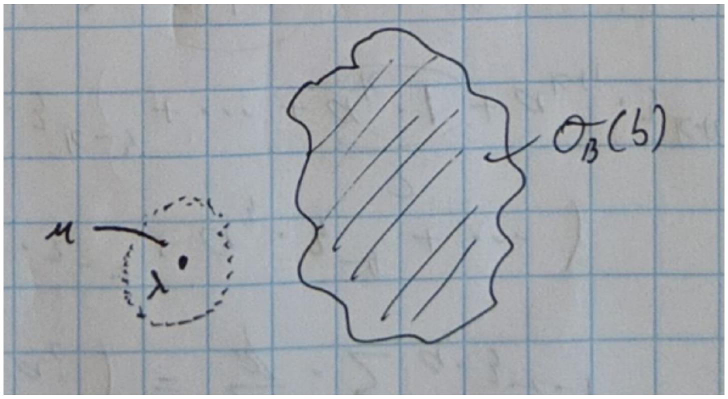
has $\left|\left|\frac{1}{\lambda} b\right|\right| < 1$ for sufficiently large λ. Therefore, $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq ||b||\}$.
To show that $\sigma_B(b)$ is closed, if $\lambda \notin \sigma_B(b)$ then $\forall \mu$ such that $||\lambda - \mu|| < \varepsilon$ we have that $\mu \notin \sigma_B(b)$.

$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b)\left\{e + \underbrace{(\mu - \lambda)(\lambda e - b)^{-1}}_{||\cdot||<1}\right\}$$

when $|\mu - \lambda| < \frac{1}{||(\lambda e - b)^{-1}||}$.
Therefore $\mathbb{C} \setminus \sigma_B(b)$ is open.

Proof of (b)

Take $\lambda \notin \sigma_B(b)$



$$
\begin{aligned}
\frac{R(b,\mu)-R(b,\lambda)}{\mu-\lambda} &= \frac{1}{\mu-\lambda}\left((\mu e-b)^{-1}-(\lambda e-b)^{-1}\right) \\
&= \frac{1}{-\mu-\lambda}(\mu e-b)^{-1}\{(\lambda e-b)-(\mu e-b)\}(\lambda e-b)^{-1} \\
&= -(\mu e-b)^{-1})(\lambda e-b)^{-1}
\end{aligned}
$$

Using continuity with $GB \ni a \mapsto a^{-1} \in GB$ in the norm, $-(\mu e-b)^{-1})(\lambda e-b)^{-1} \to -\left((\lambda e-b)^{-1}\right)^2$ as $\mu \to \lambda$.
Therefore $R^1(b,\lambda) = -(R(b,\lambda))^2$ and $R(b,\lambda)$ is analytic.

Proof of non-empty in (a)

Take $\sigma_B(b) \neq 0$, otherwise $R(b,\lambda)$ is analytic on $\mathbb{C}$ and bounded

$$(\lambda e-b)^{-1} = \frac{1}{\lambda}\left(e-\frac{1}{\lambda}b\right)^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda^{n+1}}b^n$$

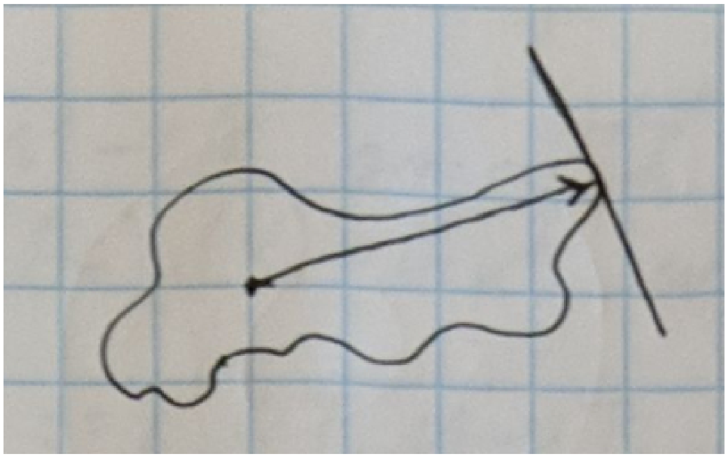
We can estimate

$$||\cdot|| \leq \frac{1}{|\lambda|\left(1-\frac{||b||}{|\lambda|}\right)} = \frac{1}{|\lambda|-||b||}$$

so $\lim_{\lambda\to\infty}||(\lambda e-b)^{-1}|| = 0$.
By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity.
If $\phi(R(b,\lambda))$ is analytic and bounded on $\mathbb{C}$, $\forall\phi \in B^*$, it follows that $\phi(R(b,\lambda)) \equiv 0$, $\forall\lambda$, $\forall\phi \in B^*$ and that $R(b,\lambda) \equiv 0$ for any λ a contradiction.

Definition: Spectral Radius

For $b \in B$, the spectral radius $r(b) = \max\{|\lambda| \,:\, \lambda \in \sigma_B(b)\}$.

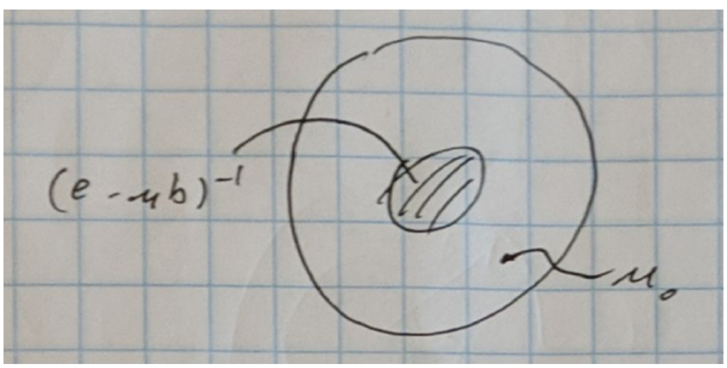


Remark

Write $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b$ is not invertible$\} = \min\{|\mu| : e - \mu b$ is not invertible$\}$with $\mu = \frac{1}{\lambda}$.

$$\underbrace{(e-\mu b)^{-1}}_{\text{analytic in } |\mu|<\frac{1}{||b||}} = \sum_{n=0} \mu^n b^n$$

converges for $|\mu| < \frac{1}{||b||}$.
Then the radius of convergence $R^{-1} = \limsup_{n\to\infty} ||b^n||^{\frac{1}{n}}$ gives us that R is equal to the largest disk where $(e-\mu b)^{-1}$ has an analytic extension. Therefore $S = \frac{1}{r(b)}$.



Suppose we have an analytic extension $f(\mu)$ beyond S.

$$f(\mu)(e-\mu b) = (e-\mu b)f(\mu) = e$$

implies that and, if $(e-\mu_0 b)$ not invertible, $f(\mu_0)(e-\mu_0 b) = \cdots = e$ a contradiction.

Theorem

$$r(b) = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} ||b^n||^{\frac{1}{n}}$$

Proof

To demonstrate existence, fix $n_0 \in \mathbb{N}$, $n = q \cdot n_0 + r$, $0 \le r < n_0$.

$$
\begin{aligned}
||b^n|| &\le ||b^{n_0}||^q \cdot ||b||^r \\
||b^n||^{\frac{1}{n}} &\le ||b^{n_0}||^{\frac{q}{n}} \cdot ||b||^{\frac{r}{n}} \\
\limsup_{n\to\infty} ||b^n||^{\frac{1}{n}} &\le ||b^{n_0}||^{\frac{1}{n_0}} \cdot 1
\end{aligned}
$$

Since $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$. Take $n \to \infty$. Write

$$\limsup_{n\to\infty} ||b^n||^{\frac{1}{n}} \le \inf_{n_0 \in \mathbb{N}} ||b^{n_0}||^{\frac{1}{n_0}} \le \liminf_{n\to\infty} ||b^n||^{\frac{1}{n}}$$

October 16, 2024

Note: Closed Subalgebras

Assume A is a closed subalgebra of B ($e \in A \subseteq B$).
Take $b \in A \subseteq B$.
Obviously, $b - \lambda e$ being invertible in A implies $b - \lambda e$ is invertible in B. We also have

$$\mathbb{C} \setminus \mathrm{sp}_A(b) \subseteq \mathbb{C} \setminus \mathrm{sp}_B(b)$$

(confer. $GA \subseteq GB$ with $\partial GA = \partial(A \cap GB)$) and, equivalently,

$$\mathrm{sp}_B(b) \subseteq \mathrm{sp}_A(b).$$

One can show similarly that

$$
\begin{array}{ccc}
\partial(\mathbb{C} \setminus \mathrm{sp}_A(b)) & \subseteq & \partial(\mathbb{C} \setminus \mathrm{sp}_B(b)) \\
= & & = \\
\partial\, \mathrm{sp}_A(b) & \subseteq & \partial\, \mathrm{sp}_B(b)
\end{array}
$$

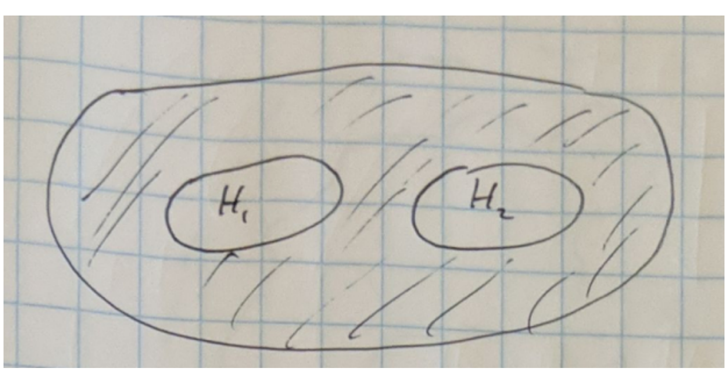
Proposition

1. $\mathbb{C} \setminus \mathrm{sp}_A(b)$ is the union of some components of $\mathbb{C} \setminus \mathrm{sp}_B(b)$.

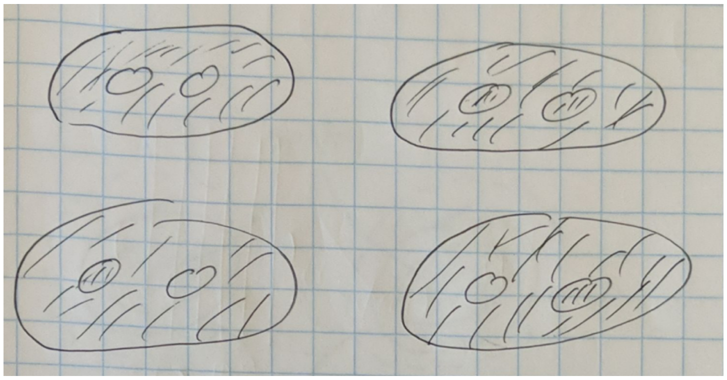
2. $\mathrm{sp}_A(b) = \mathrm{sp}_B(b) \cup \bigcup_\omega H_\omega$ where H_ω are some components of $\mathbb{C} \setminus \mathrm{sp}_B(b)$.

Example 1

Suppose $\mathrm{sp}_B(b)$ looks like

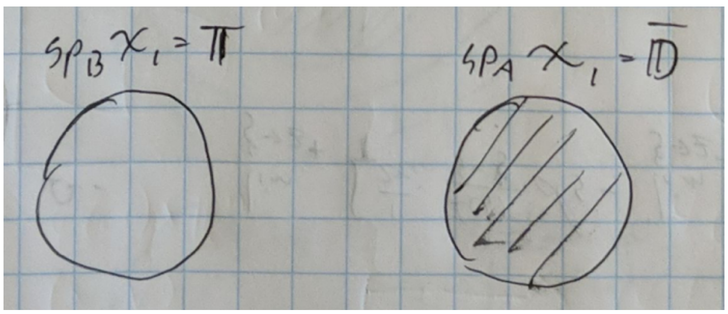


Now $\mathrm{sp}_A(b)$ can only be one of the 4 possibilities.



Example 2

$B = C(\mathbb{T})$, $A = C_+(\mathbb{T}) \simeq A(\mathbb{D})$, $\chi_1(t) = t$, $\mathrm{sp}_B\,\chi_1 = \mathbb{T}$.



Theorem: Spectral Mapping Theorem (Simple Version)

For a polynomial $p(z) = \sum_{n=0}^{N} p_n t^n$ we define $p(b) = \sum_{n=0}^{N} p_n b^n$ for $b \in B$ where $b^0 = e$.
Let p be a polynomial and $b \in B$ with B a unital Banach algebra, then $\mathrm{sp}(p(b)) = p(\mathrm{sp}(b)) := \{p(z) \,:\, z \in \mathrm{sp}(b)\}$.

Proof

For $\lambda \in \mathbb{C}$, consider $q(z) = p(z) - \lambda = c\prod_{i=1}^{N}(z-\gamma_i)$.
Now, $q(b) = p(b) - \lambda e = c\prod_{i=1}^{n}(b-\gamma_i e)$. It follows that

$$\begin{aligned}
\lambda \notin \mathrm{sp}(p(b)) &\iff p(b) - \lambda e \text{ is invertible.} \\
&\iff \overbrace{\prod_{i=1}^{N}(b-\gamma_i e)}^{\text{a commuting product}} \quad \text{is invertible.} \\
&\iff \forall i,\ b-\gamma_i e \text{ is invertible.} \\
&\iff \forall i,\ \gamma_i \notin \mathrm{sp}(b) \\
&\iff \forall z \in \mathrm{sp}(b),\ q(z) = c\prod_{i=1}^{N}(z-\gamma_i) \neq 0 \\
&\iff \forall z \in \mathrm{sp}(b),\ p(z) \neq \lambda \\
&\iff \lambda \notin p(\mathrm{sp}(b))
\end{aligned}$$

Applications

If $p(b) = 0$, then $\mathrm{sp}(b) \subseteq \{z \in \mathbb{C} : p(z) = 0\}$, because

$$\{0\} = \mathrm{sp}\,0 = \mathrm{sp}\,p(b) \overset{\text{SMT}}{=} p(\mathrm{sp}\,b).$$

It follows that if b is nilpotent, such that $b^n = 0$ for some n ($p(z) = z^2$), then $\mathrm{sp}(b) = \{0\}$.
If b is idempotent, such that $b^2 = b$ ($p(z) = z^2 - z$), then $\mathrm{sp}(b) \subseteq \{0, 1\}$.
If b is unipotent (or flip), such that $b^2 = e$, then $\mathrm{sp}(b) = \{\pm 1\}$.

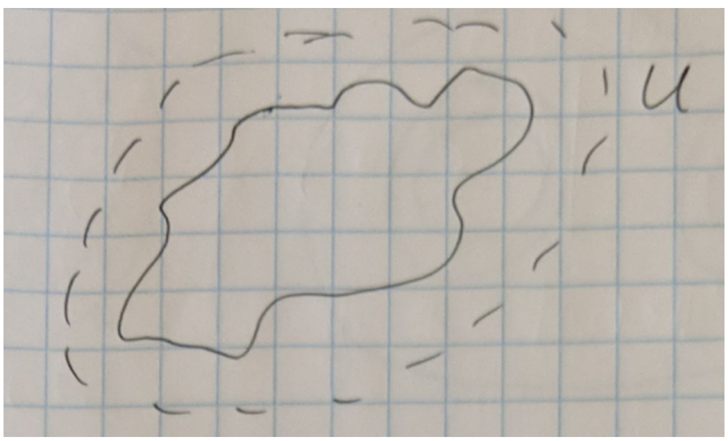
Section 1.5: Riesz Functional Calculus

Question:

Can one define $f(b)$ for $b \in B$ a unital Banach algebra for more general functions f?

Definition: Set of Functions Holomorphic on the Spectrum

For a unital Banach algebra B and $b \in B$, let $A[\mathrm{sp}(b)]$ stand for the set of all functions f which are holomorphic on some open neighborhood U of $\mathrm{sp}(b)$.



Lemma

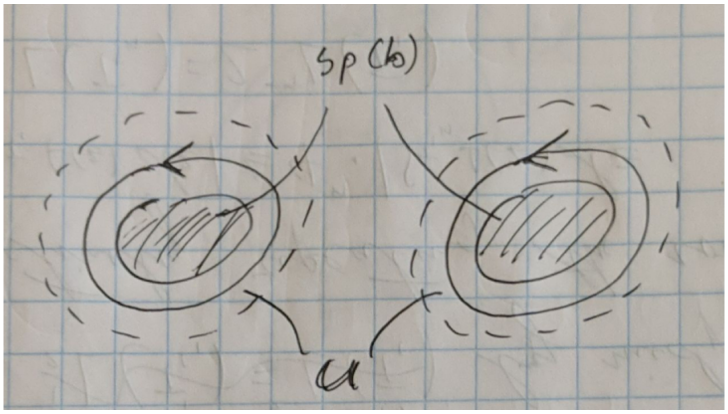
Let $f \in A[\mathrm{sp}(b)]$, i.e. $f : U \to \mathbb{C}$ holomorphic. Then there exists an open set W with (piece-)smooth boundary such that

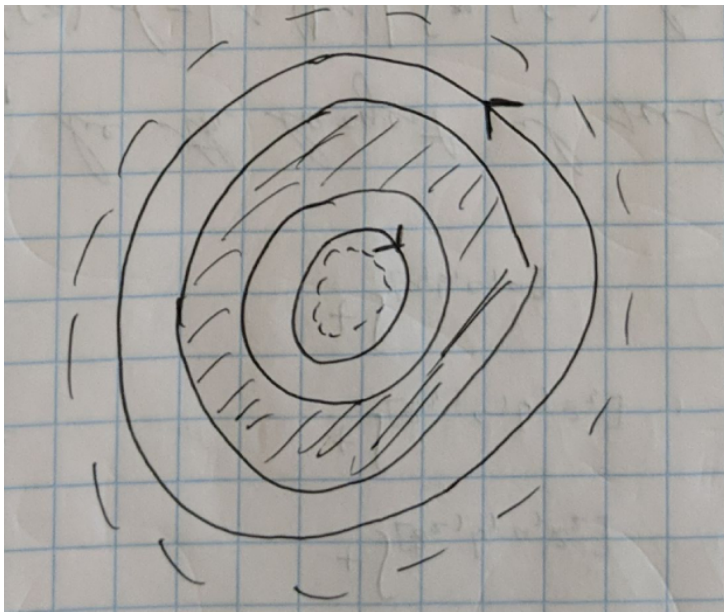
$$\mathrm{sp}(f) \subseteq W \subseteq \overline{W} \subseteq U$$

(i.e. $\partial\overline{W} \subseteq U \setminus \mathrm{sp}(b)$) and

$$\frac{1}{2\pi} \int_{\partial W} \frac{d\omega}{\omega - z} = \begin{cases} 1 & z \in \mathrm{sp}(b) \\ 0 & z \notin U \end{cases}.$$

Example





- Proof

IMAGE 7

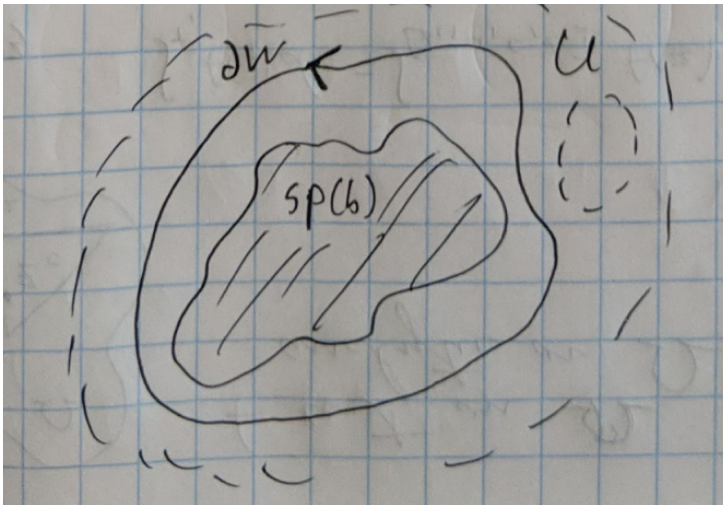
SQUARES

Definition:

Using the lemma, we define for $f \in A[\mathrm{sp}(b)]$

$$f(b) := \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda e - b)^{-1}\, d\lambda$$

(where $\mathrm{sp}(b) \subseteq W \subseteq \overline{W} \subseteq U$).
One can show that this is independent of choice of W (and also of U).
Note $f(\lambda)(\lambda e - b)^{-1}$ is holomorphic on $U \setminus \mathrm{sp}(b)$.



Remark

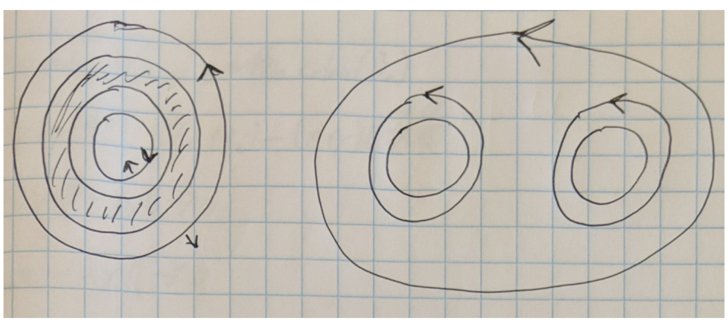
$f_1, f_2 \in A[\mathrm{sp}(b)]$ implies $f_1 + f_2 \in A[\mathrm{sp}(b)]$ and $(f_1 + f_2)(b) = f_1(b) + f_2(b)$.

Proposition

For a polynomial $f(z) = p(z) = \sum p_i z^i$, we get $f(b) = p(b) = \sum p_i b^i$.

Proof

$$\frac{1}{2\pi i}\int_{\partial W} (\lambda e - b)^{-1}\, d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=R} (\lambda e - b)^{-1}\, d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=R} \sum_{n=0}^{\infty} \frac{b^n}{\lambda^{n+1}}\, d\lambda = e$$



Therefore, $p(b) = \frac{1}{2\pi i}\int_{\partial W} p(b)(\lambda e - b)^{-1}\, d\lambda$, and

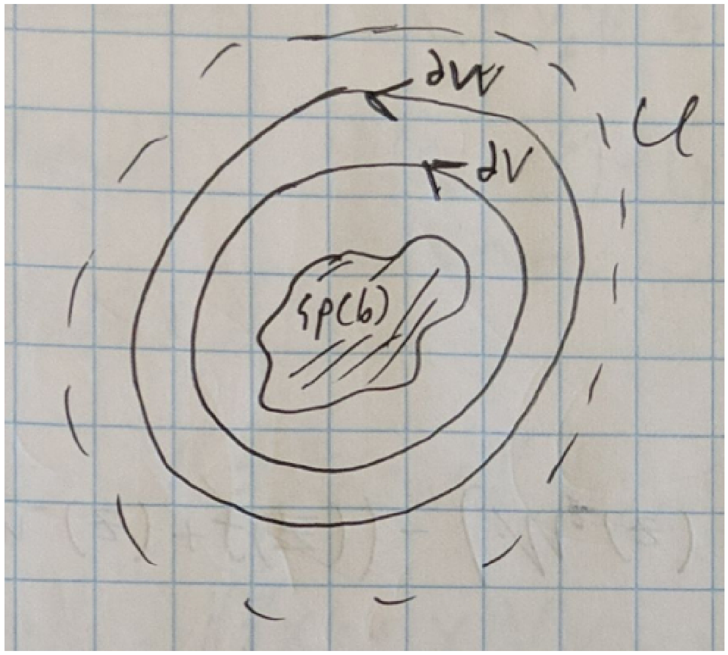
$$\begin{aligned} f(b) - p(b) &= \frac{1}{2\pi i}\int_{\partial W} (f(\lambda)e - p(b))(\lambda e - b)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial W} \sum_{n=0}^{N} \underbrace{(\lambda^n e - b^n)}_{(\lambda^{n-1}e + \cdots + b^{n-1})(\lambda e - b)} (\lambda e - b)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial W} \text{"polynomial in } \lambda, b\text{"}\, d\lambda = 0 \end{aligned}$$

Proposition

If $f_1, f_2 \in A[\mathrm{sp}(b)]$, then $f_1 f_2 \in A[\mathrm{sp}(b)]$.

$$(f_1 f_2)(b) = f_1(b) \cdot f_2(b)$$

Proof



We assume ∂V is inside ∂W.

$$
\begin{aligned}
f_1(b) &= \frac{1}{2\pi i}\int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1}\, d\lambda \\
f_2(b) &= \frac{1}{2\pi i}\int_{\partial V} f_2(\xi)(\xi e - b)^{-1}\, d\xi
\end{aligned}
$$

Then

$$
f_1(b)f_2(b) = \frac{1}{(2\pi i)^2}\int_{\partial W}\int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1}(\xi e - b)^{-1}\, d\xi d\lambda
$$

Recall that

$$
(\lambda e - b)^{-1}(\xi e - b)^{-1} = (\lambda e - b)^{-1}\left[\frac{(\lambda e - b) - (\xi e - b)}{\lambda - \xi}\right](\xi e - b)^{-1} = \frac{(\xi e - b)^{-1} - (\lambda e - b)^{-1}}{\lambda - \xi}
$$

Therefore

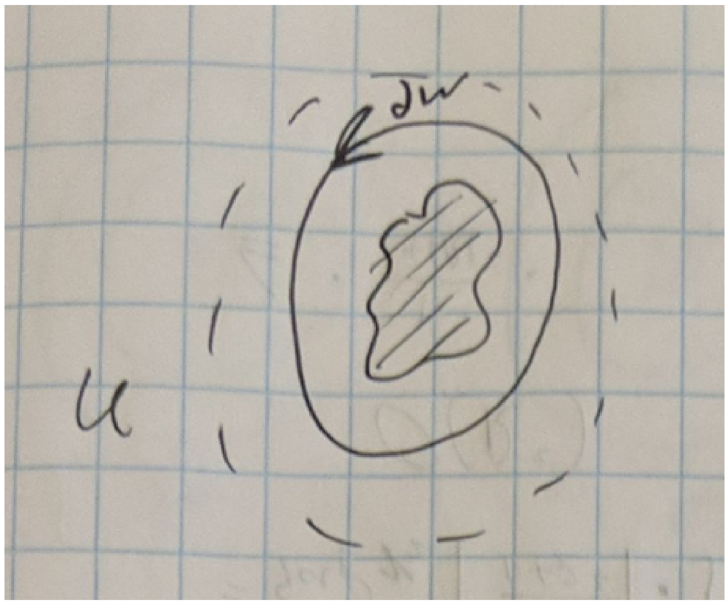
$$
\begin{aligned}
f_1(b)f_2(b) &= \frac{1}{(2\pi i)^2}\int_{\partial V}\int_{\partial W} f_1(\lambda)f_2(\xi)(\xi e - b)^{-1}\frac{1}{\lambda - \xi}\, d\lambda d\xi - \frac{1}{(2\pi i)^2}\int_{\partial W}\int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1}\frac{1}{\lambda - \xi}\, d\xi d\lambda \\
&= \frac{1}{(2\pi i)^2}\int_{\partial V} f_2(\xi)(\xi e - b)^{-1}\underbrace{\int_{\partial W}\frac{f_1(\lambda)}{\lambda - \xi}\, d\lambda}_{\equiv f_2(\xi)} d\xi - \frac{1}{(2\pi i)^2}\int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1}\underbrace{\int_{\partial V}\frac{f_2(\xi)}{\lambda - \xi}\, d\xi}_{\equiv 0} d\lambda \\
&= \frac{1}{2\pi i}\int_{\partial V} f_2(\xi)f_1(\xi)(\xi e - b)^{-1}\, d\xi \\
&= (f_1 f_2)(b)
\end{aligned}
$$

October 21, 2024

Recall

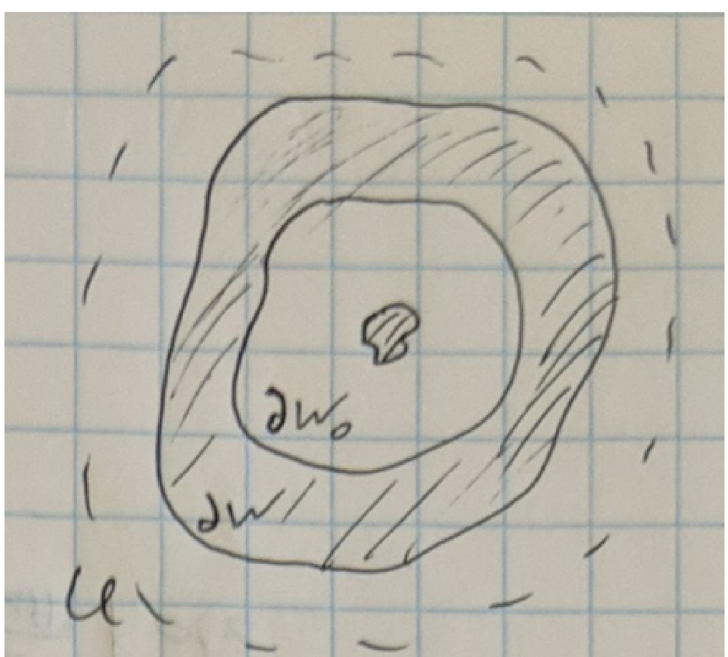
$f \in A[\operatorname{sp} b]$, $f : U \to \mathbb{C}$, $\operatorname{sp}(b) \subseteq U$ open. Define

$$(1) \quad f(b) = \frac{1}{2\pi i} \int_{\partial W} \underbrace{f(\lambda)(\lambda e - b)^{-1}}_{\text{analytic in } U \setminus \operatorname{sp} b} \, dz$$



with $\operatorname{sp} b \subseteq W \subseteq \overline{W} \subseteq U$ and ∂W piecewise smooth.
From the above lemma, applied to W, we get W_0 such that $\operatorname{sp} b \subseteq W_0 \subseteq \overline{W}_0 \subseteq W \subseteq \overline{W} \subseteq U$. Then

$$(2) \quad \frac{1}{2\pi i} \int_{\partial W_0} f(\lambda)(\lambda e - b)^{-1} \, dz$$



with $V = W \setminus W_0$, $\partial V = \partial W \cup \partial W_0$.

$$(1) - (2) = \frac{1}{2\pi i} \int_{\partial V} \underbrace{f(\lambda)(\lambda e - b))^{-1}}_{\text{holomorphic on } V} dz = 0$$

and $V \subseteq \overline{V} \subseteq U \setminus \mathrm{sp}(b)$.

Results

$f_1, f_2 \in A[\mathrm{sp}\, b] \implies f_1 + f_2 \in A[\mathrm{sp}\, b]$
$f_1(b) + f_2(b) = (f_1 + f_2)(b)$.
For f polynomial, $\sum_{n=0}^{N} f_n t^n$, $f(b) = \sum_{n=0}^{N} f_n b^n$.

Proposition

$f_1(b) f_2(b) = (f_1 f_2)(b)$.

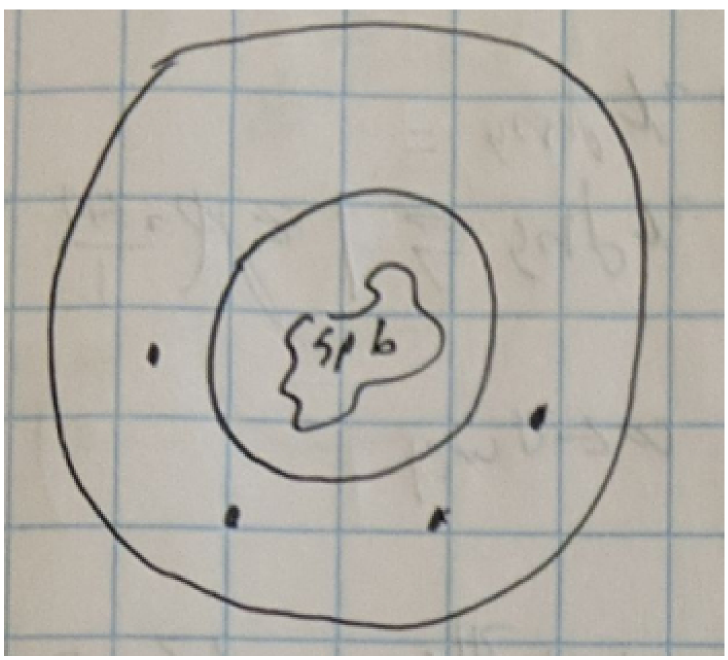
Theorem: Spectral Mapping Theorem

Let $b \in B$ and $f \in A[\mathrm{sp}\, b]$. Then $\mathrm{sp}(f(b)) = f(\mathrm{sp}\, b) := \{f(z) \, : \, z \in \mathrm{sp}\, b\}$.

Proof

1. take $\mu \notin f(\mathrm{sp}\, b)$.

   Then $\mu \notin f(z)$, $\forall z \in \mathrm{sp}\, b$ and $\mu - f(z) \neq 0$.

   Therefore, there exist an open $U_1 \supseteq \mathrm{sp}(b)$, $U_1 \subseteq U$, such taht $\mu - f(z) \neq 0$, $\forall z \in U_1$.
   Define $g(z) = \frac{1}{u - f(z)}$ holomorphic on U_1, and

$$g(z) \cdot (\mu - f(z)) = 1 \implies g(b) \cdot (\mu e - f(b)) = e$$

by the previous proposition and the polynomial result. So $\mu e - f(b)$ is invertible, and $\mu \notin \mathrm{sp}(f(b))$.

- Remark

$$(\mu e - f(b))^{-1} = \frac{1}{2\pi i}{}_{\partial W_1} \frac{1}{\mu - f(z)} (ze - b)^{-1}\, dz$$

for $\operatorname{sp} b \subseteq W_1 \subseteq \overline{W}_1 \subseteq U_1$.

- take $\mu \notin \operatorname{sp}(f(b))$ and, for contradiction, assume $\mu \in f(\operatorname{sp} b)$.

  Then $\mu e - f(b)$ is invertible, $\mu = f(\lambda)$ for some $\lambda \in \operatorname{sp} b$.

- Idea

$$\mu e - f(b) = f(\lambda)e - f(b) = (\lambda e - b) \cdot g_\lambda(b)$$

  We define

$$g_\lambda(z) = \begin{cases} \frac{f(\lambda)e - f(z)}{\lambda - z} & z \in U \supseteq \operatorname{sp}(b) \\ f'(\lambda) & z = \lambda \end{cases}$$

  such that $g_\lambda(z)$ is holomorphic on U. Therefore $g_\lambda(b) \in B$,

$$(\lambda - z) g_\lambda(z) = f(\lambda) - f(z), \quad \forall z \in U$$

  and $(\lambda e - b) g_\lambda(b) = f(\lambda)e - f(b) = g_\lambda(b)(\lambda e - b)$. Since this is invertible, $(\lambda e - b)$ is left and right invertible.

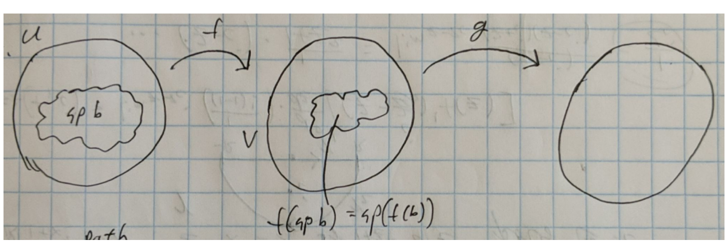
Remark

$$g_\lambda(b) = \frac{1}{2\pi i} \int_{\partial_W} \frac{f(\lambda) - f(z)}{\lambda - z} (ze - b)^{-1}\, dz$$

Theorem: Composition of Functions

Let $b \in B$ unital, $f \in A[\operatorname{sp} b]$, and $g \in A[\operatorname{sp}(f(b))] = A[f(\operatorname{sp} b)]$.
Then $h = g \circ f \in A[\operatorname{sp} b]$ and $h(b) = g(f(b))$.

Remark



f is an open mapping and maps U to the open set $V \supseteq \operatorname{sp}(f(b))$.

Applications

- Exponentials

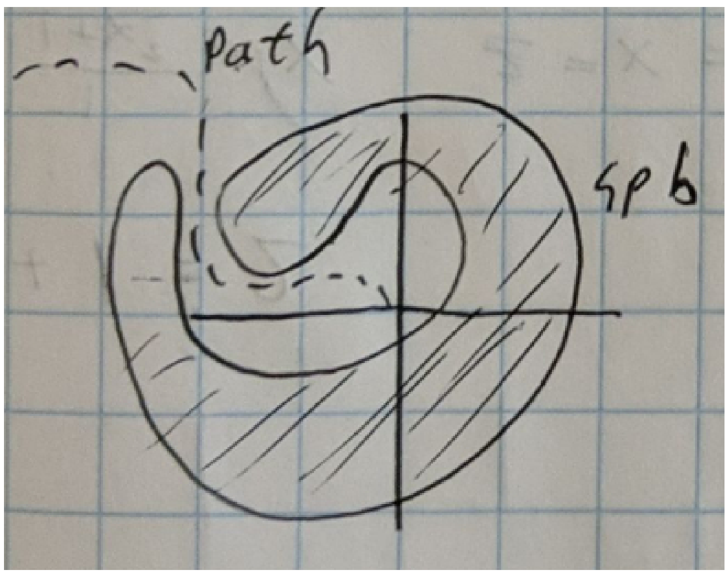
$$\exp(b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} = \frac{1}{2\pi i} \int_{|z|=R} e^z (ze-b)^{-1}\, dz$$

- Logarithms

  $\log b,\ b \in B$ under the assumption that

  - $0 \notin \operatorname{sp} b$
  - There exists a path connecting 0 to ∞ in $\mathbb{C} \setminus \operatorname{sp} b$.

  This gives us that $\log z$ is analytic on $U \supseteq \operatorname{sp} b$.



  $\mathbb{C}\setminus$path is simply connected, so there exists an analytic $\log z$ on $\mathbb{C}\setminus$path.

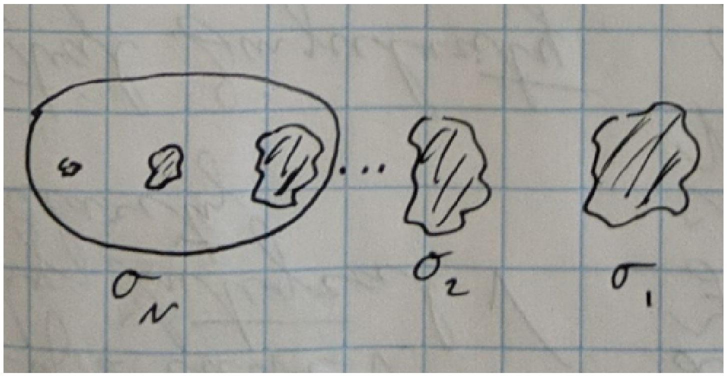
- if $\log b$ is well-defined, then $\exp(\log b)) = b$ (via composition)
- likewise, one can define powers $f(z) = z^{\alpha}$ ($\alpha \in \mathbb{C}$)

Application: Spectral Idempotents (Riesz Idempotents)

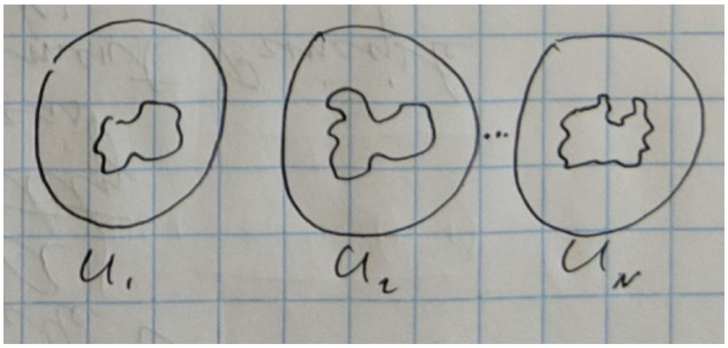
p is idempotent if $p^2 = p$.
Assume that $b \in B$ and that $\operatorname{sp} b$ is not connected.

$$\operatorname{sp} b = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_N$$

with σ_i closed and disjoint subsets of $\operatorname{sp} b$.



Now let $U_1, \dots, U_n$ be open neighborhoods of $\sigma_1, \dots, \sigma_n$ which are themselves disjoint.



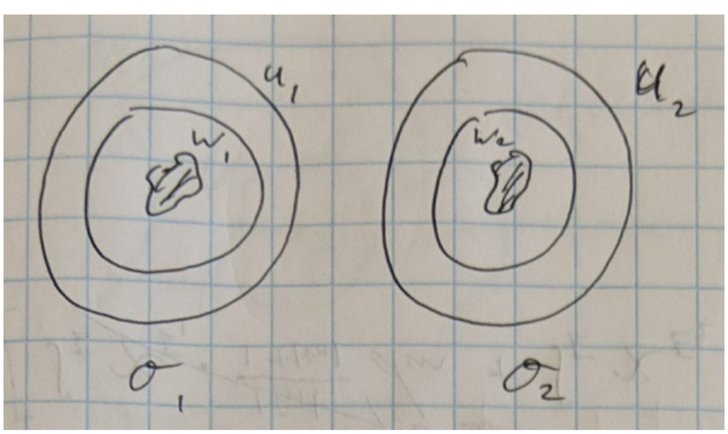
Write $U = U_1 \cup \cdots \cup U_n \supseteq \operatorname{sp} b$, and consider

$$\chi_i(x) = \begin{cases} 1 & x \in U_i \\ 0 & x \in U_j,\ j \neq i \end{cases}$$

Then χ_i is analytic on $U \supseteq \operatorname{sp}(b)$.
Put $p_i = \chi_i(b)$ the spectral or Riesz idepotents.

Properties / Remarks

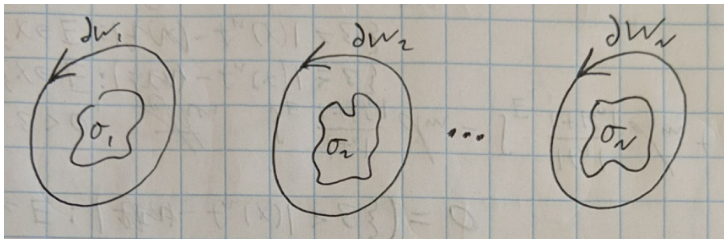
- $p_i^2 = p_i$ because $(\chi_i)^2 = \chi_i$.
- $e = p_1 + \cdots p_N$, mutually orthogonal such that $p_i p_j = 0$, $\forall i \neq j$, because $\chi_1 + \cdots + \chi_N = 1$ and $\chi_i \chi_j = 0$.
- $p_i b = b p_i$, because $\chi_i f = f \chi_i$ for $f(z) = z$.
- $p_i = \frac{1}{2\pi i} \int_{\partial W} \chi_i(z)(ze - b)^{-1}\, dz$ where $\operatorname{sp} b \subseteq W \subseteq \overline{W} \subseteq U$.



$W_i = W \cap U_i$, $W_1 \cup \cdots W_N = W$. Therefore

$$p_i = \frac{1}{2\pi i} \sum_{j \neq 1}^{N} \int_{\partial W_j} \chi_i(z)(ze-b)^{-1}\, dz = 0, \quad i \neq j$$

Then $p_i = \frac{1}{2\pi i} \int_{\partial W_i} (ze-b)^{-1}\, dz$.



- Write

$$b = (p_1 + p_2 + \cdots + p_N) b (p_1 + p_2 + \cdots + p_N) = p_1 b p_1 + p_2 b p_2 + \cdots + p_N b p_N$$

since $p_i b p_j = b p_i p_j = 0$.

- For an idempotent $p \neq 0$,

$$B_p = \{pap \,:\, a \in B\}$$

and, therefore, B_p has a unit element p.

Lemma

Assume $b \in B$ with Riesz idempotents $p_1, \ldots, p_N \neq 0$.
Then b is invertible if and only if $p_i b p_i$ is invertible in B_{p_i} for all i.

Proof

$b^{-1} = c$, $bc = e$, then

$$\begin{aligned}(p_1 + \cdots + p_N) b (p_1 + \cdots + p_N) c &= e \\ \sum p_i b (p_i p_i) c &= e \\ (p_i b_i)(p_i c p_i) &= p_i\end{aligned}$$

Suppose $p_i b p_i$ invertible in B_{p_i}. Then $p_i b p_i c = p_i$, $c_i = p_i c p_i$ and $b^{-1} = c = \sum_{i=1}^N p_i c_i p_i$.

Remark

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_N \end{pmatrix} \quad P_1 = \begin{pmatrix} I & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} P_2 = \begin{pmatrix} 0 & & & \\ & I & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Therefore B invertible if and only if B_i are invertible. $B_i \cong P_i B P_i$.

$$\begin{pmatrix} 0 & & & \\ & B_i & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

October 23, 2024

Lemma

Let $b \in B$ and $p_1, \dots, p_n \in B$ satisfying $p_i^2 = p_i$, $p_i p_j = 0$ $(i \neq j)$, $p_1 + \cdots p_n = e$, $bp_i = p_i b$.
Then b is invertible in B if and only if for each i, $p_i b p$ are invertible in B_{p_i} and

$$\mathrm{sp}_B(b) = \bigcup_{i=1}^{N} \mathrm{sp}_{B_{p_i}}(p_i b p_i)$$

where $B_{p_i} = \{p_i a p_i : a \in B$ is a unital Banach algebra with unit p_i.

Theorem

Let $p_1, \dots, p_N$ be spectral idempotents of b with respect to $\mathrm{sp}(b) = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_N$ (closed and disjoint).
If $\sigma_1, \dots, \sigma_N \neq \varnothing$, then $p_1, \dots, p_N \neq 0$ and $\mathrm{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$.
Note: if $\sigma_i = \varnothing$ then

$$p_i = \chi_{U_i}(b) = \frac{1}{2\pi i} \int_{\partial W_i} |\underbrace{ze - b}_{\text{analytic}}|^{-1} \, dz.$$

Proof

Without loss of generality, we may assume $p_1, \dots, p_M \neq 0$ $(M \geq 1)$ and $p_{M+1} = \cdots = p_N = 0$.
Then by the above lemma $p_1 + \cdots + p_M = e$ and

$$\mathrm{sp}_B(b) = \bigcup_{i=1}^{M} \mathrm{sp}_{B_{p_i}}(p_i b p_i)$$

Assuming $\mathrm{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ is proven for $j = 1, \dots, M$, then

$$\mathrm{sp}_B(b) = \bigcup_{i=1}^{M} \sigma_i = \bigcup_{i=1}^{N} \sigma_i$$

and therefore that $M = N$.
To prove that $\mathrm{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ for each $p_i \neq 0$,

$$\mathrm{sp}_{B_{p_i}}(p_i b p_i) = \{\lambda \in \mathbb{C} : p_i(b - \lambda)p_i + e - p_i \text{ not invertible in } B\}$$

For fixed λ, $f_\lambda(z) = \chi_i(z)(z - \lambda)\chi_i(z) + (1 - \chi_i(z))$ is analytic in a neighborhood of $\mathrm{sp}(b)$.

$$f_\lambda(b) = p_i(b - \lambda e)p_i + (1 - p_i)$$

Then $\lambda \in \mathrm{sp}_{B_{p_i}}(p_i b p_i)$ if and only if $f_\lambda(b)$ is not invertible in B.
Equivalently that $0 \in \mathrm{sp}(f_\lambda(b))$ or, by spectral mapping theorem, $0 \in f_\lambda(\mathrm{sp}\, b)$.
This is further equivalent to there existing some $\xi \in \mathrm{sp}\, b : 0 = f_\lambda(\xi)$

$$f_\lambda(z) = \begin{cases} 1 & z \in \sigma_j \subseteq U_j,\ i \neq j \\ z - \lambda & z \in \sigma_i \end{cases}$$

That is, if $\xi \in \mathrm{sp}\, b : \xi \in \sigma_i$ and $\xi = \lambda$ or, simply, $\lambda \in \sigma_i$.

Chapter 2: Commutative Banach Algebras

Section 2.1: Homomorphisms, Ideals and Quotient Algebras.

B need not be commutative.

Definition: Banach Algebra Homomorphisms

$\phi : A \to B$ is a Banach algebra homomorphism if it is linear, multiplicative and bounded.

Definition: Banach Algebra Ideal

A (two-sided) ideal J of a Banach algebra is a linear subspace $J \subseteq A$ such that $\forall\, a \in A,\ \forall\, j \in J,\ aj, ja \in J$.

Remark

If $\phi : A \to B$ is a Banach algebra homomorphism then $\ker \phi$ is a closed two-sided ideal of A.

Proof

Put $J \in \ker \phi,\ a \in A,\ j \in J$. Then $\phi(j) = 0,\ \phi(aj) = \phi(a)\phi(j) = 0 = \phi(j)\phi(a) = \phi(ja)$ and $aj, ja \in J$.

Definition: Quotient Algebra

If J is a closed, two-sided ideal of A ($J \neq A$), then A/J is a Banach algebraA/J is a Banach algebra $[a] = a + J$.
A/J is a vector space, a normed space (J closed) with $||[a]_J|| = \inf_{j \in J} ||a + J||$, and a Banach space because A is complete.
$[a_1] + [a_2] = [a_1 + a_2]$ and $[a_1] \cdot [a_2] = [a_1 \cdot a_2]$

$$(a_1 + j_1)(a_2 + j_2) = a_1 a_2 + \underbrace{a_1 j_2 + a_2 j_1 + j_1 j_2}_{\in J}$$

Definition: Quotient Map

Take $\pi : A \to A/J$ by $a \mapsto [a]$.
This is a Banach algebra homomorphism which is surjective with $\ker \pi = J$.

Proposition

Let $\phi : A \to B$ be a Banach algebra homomorphism and $J \subseteq \ker \phi$ a closed, two-sided ideal of A.
Then there exists a Banach algebra homomorphism $\phi^J : A/J \to B$ such that $\phi = \phi^J \circ \pi$

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & B \\ & A/J \arrow[ur, "\phi^J"'] & \end{tikzcd}$$

Write $\phi^J([a]_J) = \phi(a)$, and $[a_1] = [a_2]$ implies $a_1 - a_2 \in J \subseteq \ker\phi$ and subsequently that $\phi(a_1) = \phi(a_2)$.

Remark

$J = \{0\}$ and $J = A$ are always closed, two-sided ideals of A.

Examples

- $A = \mathbb{C}^{n\times n}$. Only ideals are $\{0\}$ and A.
- $A = L(X)$ (continuous operators) for X a Banach space. Then at least $\{0\}$, $K(X)$ (compact operators), and A are ideals.
- X a separable hilbert space. Only $\{0\}$, $K(X)$ and A.
- $A = \mathbb{C}^{n\times n}_{\text{upper}}$ upper triangular matrices. Then there are many (one sided) ideals for $n = 2$.
- $A = C(X)$ for X compact Hausdorf spaces. Then every closed set $E \subseteq X$ generates a closed ideal

$$J_E = \{f \in C(x) : f|_E \equiv 0\}$$

In particular, $E = \{x_0\}$, $J_{x_0} = \{f \in C(X) \,:\, f(x_0) = 0\}$, $\dim(A/J_{x_0}) = 1$ implies $A/J_{x_0} \cong \mathbb{C}$.

Remark

Every closed (2-sided) ideal is a closed subalgebra of A but not vice versa.
For a set $S \subseteq A$, let $J = \operatorname{clos}\,\operatorname{id}_A(S)$ be the smallest closed 2-sided ideal containing S (i.e. the ideal generated by S or the intersection of all ideals containing S). One can show that

$$J = \operatorname{clos}_A\left\{\sum_{i=1}^{N} a_i j_i b_i \,:\, a_i, b_i \in A,\ j_i \in S\right\}$$

Section 2.2: Maximal Ideals and Multiplicative Linear Functionals

From now on, B is a unital, commutative Banach algebra.

Definition: Multiplicative Linear Functional

A multiplicative linear functional on B is a linear map $\phi : B \to \mathbb{C}$ such that $\phi(ab) = \phi(a)\phi(b)$ ($\phi \not\equiv 0$).

Proposition

A multiplicative linear functional on B is bounded. In fact $\phi \in B^*$, $||\phi|| = 1$, $\phi(e) = 1$.

Proof

$\phi \not\equiv 0$ means that there exists $a \in B$ such that $\phi(a) \neq 0$.
Then $\phi(e)\phi(a) = \phi(ea) = \phi(a)$ so $\phi(e) = 1$ and consequently that $||\phi|| \geq 1$.

If $|\phi(a)| \le ||a||$, then $||\phi|| \le 1$. If this were not the case,

$$|\phi(a)| > ||a|| \quad \iff \quad \left|\left|\frac{a}{\phi(a)}\right|\right| < 1$$

and $e - \frac{a}{\phi(a)}$ is invertible. Call the inverse b. Then

$$b\left(e - \frac{a}{\phi(a)}\right) = e \quad \implies \quad \phi(b)\ \underbrace{\phi\left(e - \frac{a}{\phi(a)}\right)}_{=\phi(e)-\frac{1}{\phi(a)}\phi(a)=0} = \phi(e) = 1$$

which is a clear contradiction.

Definition: Maximal Ideal

A (two-sided) ideal I of B is called maximal if

- $I \neq B$ (I is a proper ideal)
- if J is another ideal of B such that $I \subseteq J \subseteq B$, then either $I = J$ or $J = B$.

Proposition

A maximal ideal I is closed and B/I is a field.

Proof (Closed)

We have $I \subseteq \overline{I} \subseteq B$ with $\overline{I}$ an ideal. Since I is maximal, either $I = \overline{I}$ or $\overline{I} = B$. But then $e \in \overline{I}$, and there exists $a \in I$ such that $||a - e|| < 1$. Then $a = e + (a - e)$ is invertible and for each $b \in B$, $b = ba^{-1}a \in I$ and $I = B$ a contradiction.

October 28, 2024

Recall: Multiplicative Linear Functionals

$\phi : B \to \mathbb{C}$

- linear
- $\phi(ab) = \phi(a)\phi(b)$, $\phi \not\equiv 0$.

Then $\phi(e) = 1$, $||\phi|| = 1$

Recall: Maximal Ideals

I is a maximal ideal of B if

- $I \underset{\neq}{\subseteq} B$
- If J is an ideal with $I \subseteq J \subseteq B$, then either $I = J$ or $J = B$.

Proposition

Every maximal ideal is closed and, in a commutative Banach algebra B, B/I is a field.

Proof (Field)

We know that B/I is a Banach algebra. We need to show that every nonzero $[a] \in B/I$ is invertible.
Consider $[a] \in B/I$ with $[a] \neq 0$. Then $a \notin I$. Define

$$J = \{i + ax \,:\, i \in I, x \in B\}$$

Then J is a linear subspace and an ideal in B since for any $y \in B$

$$\begin{aligned} y(i + ax) &= yia(yx) \in J \\ (i + ax)y &= \underbrace{iy}_{\in I} + a(xy) \in J \end{aligned}$$

Since $I \subseteq J \subseteq B$ and I is maximal, it cannot be that $I = J$ since $0 + a \cdot e \in J$ implies $a \in I$ a contradiction.
If $J = B$, then $e \in J$ and $e = i + ax$ for some $i \in I$ and $x \in B$. Therefore, in the quotient,

$$[e] = [a] \cdot [x] = [x] \cdot [a]$$

and $[a]$ is invertible in B/I.

Theorem: Gelfand/Mazur

Any (complex) Banach algebra which is a field is isomorphic to $\mathbb{C}$.

Proof

Let A be a Banach algebra which is a field with unit $e \in A$, and consider the map

$$\Lambda : \mathbb{C} \ni \lambda \mapsto \lambda e \in A$$

This Banach algebra homomorphism is isometric ($||\lambda e|| = |\lambda|$) and injective.
If Λ is surjective, then it is a Banach algebra isomorphism.
Take $a \in A$. We know that $\mathrm{sp}(a) \neq \varnothing$, therefore $\exists \lambda \in \mathbb{C}$ such that $\lambda e - a$ is not invertible.
It follows that $\lambda e - a = 0$ and, consequently, $a = \lambda e = \Lambda(\lambda)$.

Corollary

Let B be a unital, commutative Banach algebra and I be a maximal ideal. Then $B/I \cong \mathbb{C}$.

Theorem: 1-1 Correspondence Between Maximal Ideals and Multiplicative Linear Functionals

Let B be a unital, commutative Banach algebra.

1. If ϕ is a multiplicative linear functional on B, then $\ker \phi$ is a maximal ideal in B.

2. If I is a maximal ideal in B, then there exists a unique multiplicative linear functional ϕ such that $\ker \phi = I$.

Proof of 1

If ϕ is a multiplicative linear functional, it is bounded and $\ker\phi$ is a closed, two-sided ideal.
We know $I \subsetneq B$ because $\phi \not\equiv 0$ ($\phi(e) = 1$).
We have that $B = \ker\phi + \mathbb{C} \cdot e$ becasue

$$b = \underbrace{b - \phi(b) \cdot e}_{\in \ker\phi} + \phi(b) \cdot e$$
$$\phi(b - \phi(b) \cdot e) = \phi(b) - \phi(b) \cdot \phi(e) = 0$$

and therefore $\dim B/I = 1$.
If $I \subseteq J \subseteq B$, then $J/I \subseteq B/I$ and either $\dim(J/I) = 0$ ($J = I$) or $\dim(J/I) = 1$ ($J = B$).

Proof of 2

Let I be a maximal ideal in B.
By Gelfand/Mazur (corollary) $B/I \cong \mathbb{C}$, so there exists a Banach algebra isomorphism $\psi : B/I \to \mathbb{C}$.

$$\begin{tikzcd} B \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathbb{C} \\ & B/I \arrow[ur, "\psi"'] & \end{tikzcd}$$

where π and ψ are Banach algebra homomorphisms.
Put $\phi = \psi \circ \pi : B \to \mathbb{C}$ by $\phi(b) = \psi([b])$. Then ϕ is a Banach algebra homomorphism that is also a multiplicative linear functional with $\phi(e) = 1$. Since ψ is injective,

$$\ker\phi = \ker\pi = I$$

Suppose $\ker\phi_1 = \ker\phi_2 = I$. Then

$$b - \phi_1(b) \cdot e \in \ker\phi_1 = I$$
$$b - \phi_2(b) \cdot e \in \ker\phi_2 = I$$

and $(\phi_1(b) - \phi_2(b))e \in I$. But a proper ideal cannot contain invertible elements. Therefore $\phi_1(b) = \phi_2(b)$.
Since this holds for all b, $\phi_1 \equiv \phi_2$.

Definition: Maximal Ideal Space

For a unital (complex) commutative Banach algebra B, $M(B)$ denotes the maximal ideal space (i.e. the set of all multiplicative linear functionals on B).

Remark

$M(B) \subseteq B^*$.
$M(B) \subseteq \{\phi \in B^* \, : \, ||\phi|| \leq 1\}$.
The multiplicative linear functionals are a proper subset of bounded linear functionals.

Example

$B = C(X)$ (the set of continuous functions $f : X \to \mathbb{C}$) for X a compact Hausdorff space.
Every $x_0 \in X$ determines a multiplicative linear functional $\phi_{x_0} : B \to \mathbb{C}$ by $\phi_{x_0}(f) = f(x_0)$ for $f \in B$.

The corresponding maximal ideals are of the form $I_{x_0} = \ker\phi_{x_0} = \{f \in C(x) : f(x_0) = 0\}$. Then

$$C(X) = I_{x_0} \dot{+} \mathbb{C} \cdot e, \quad e(x) \equiv 1$$
$$C(X)/I_{x_0} \cong \mathbb{C}$$

Therefore $\phi_{x_0} \in M(C(X))$. In fact one can show that $M(C(X)) = \{\phi_x : x \in X\}$. So $M(C(X)) \cong X$.

Remark

$M(C(X)) \subseteq C(X)^*$.
$C(X)^*$ is isomorphic to the set of all complex Boreal measures on X by

$$\phi(f) = \int_X f(x)\, d\mu(x)$$

with bounded linear functional $\mu = \delta_{x_0} \rightsquigarrow \delta_{x_0} \in M(C(X))$.

Section 2.3: Gelfand Theory and Gelfand Transform

Setting

Unital commutative (complex) Banach algebra B.

Lemma

Every proper (two-sided) ideal I_0 of B is contained in some maximal ideal of B.

Proof

Zorn's lemma applied to the collection S of all proper ideals.

Lemma

Every non-invertible element $a \in B$ is contained in at least one maximal ideal.

Proof

Consider $a \in B$ and $I_0 \in \{ax : x \in B\}$ an ideal ($y(ax) = a(yx) \in I_0$ and $(ax)y = a(xy) \in I_0$).
I_0 is proper, otherwise $I_0 = B$, $e \in I$ and $ax = xa = e$ a contradiction.
Then, by the previous lemma, $a = ae \in I_0 \subset I$ for some maximal ideal I.

Theorem: Gelfand Theory

Let B be a unital commutative Banach algebra and $b \in B$ arbitrary. Then b is invertible in B if and only if $\phi(b) \neq 0$, $\forall \phi \in M(B)$.

Proof

($\Longrightarrow$) if b is invertible,

$$1 = \phi(e) = \phi(b^{-1}b) = \phi(b^{-1}) \cdot \phi(b)$$

so $\phi(b) \neq 0$ for all ϕ.
($\Longleftarrow$) if b is not invertible, then there exists $\phi \in M(B)$ such that $\phi(b) = 0$.

If b is not invertible, then b is contained in some maximal ideal I and $I = \ker\phi$ for some $\phi \in M(B)$.
Therefore $b \in I = \ker\phi$ implies $\phi(b) = 0$.

Definition/Notation: Gelfand Transform

The Gelfand transform of an element $b \in B$ is the function $\hat{b} : M(B) \to \mathbb{C}$ defined by $\hat{b}(\phi) := \phi(b)$.

Remark

$\widehat{a+b} = \hat{a} + \hat{b}$
$\widehat{ab} = \hat{a} \cdot \hat{b}$
$\sup_{\phi\in M(B)} |\hat{a}(\phi)| = \sup_{\phi\in M(B)} |\phi(a)| \le ||a||$.
Later we will consider $M(B)$ with topology (a compact Hausdorff space).

Definition: Gelfand Transform of B

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

It is a Banach algebra homomorphism.
Gelfand's theorem states that b is invertible in B if and only if $\Lambda(b)$ is invertible in $C(M(B))$.

Note

b is invertible if and only if $\phi(b) = \hat{b}(\phi) \neq 0$, $\forall \phi \in M(B)$.
Equivalently, $\hat{b}$ is invertible in $C(M(B))$.
A continuous finctional is invertible within the set of continuous functions if and only if it is non-zero everywhere.

Purpose of Gelfand Theory

Invertibility in B corresponds to invertibility in $C(M(B))$.
We need to determine $M(B)$.

Remark

If $B = C(X)$ (X compact Hausdroff), then $M(B) = M(C(X))$ is homeomorphic to X.
$M(B) \cong X$ (homeomorphic) implies that $C(M(B)) \cong C(X) = B$ isometric Banach algebra isomorphisms.

October 30, 2024

Recall

$M(B)$ the multiplicative linear functionals or the maximal ideal space.
b invertible if and only if $\phi(b) \neq 0$, $\forall \phi \in M(B)$.
$\hat{b} : M(B) \to \mathbb{C}$ where $\hat{b}(\phi) = \phi(b)$ the Gelfand transform of b.
$\Lambda : B \to C(M(B))$ where $b \mapsto \hat{b}$ is the Gelfand transform of B.

Section 2.4: The Topology of the Maximal Ideal Space

Since $M(B) \subseteq \{\phi \in B^* \,:\, ||\phi|| \le 1\} \subseteq B^*$, $M(B)$ is a topological space with the subspace topology with respect to the weak*-topology of B^*.
A base for the topology in $M(B)$ is given by

$$U_{\varepsilon;b_1,\dots,b_n}[\phi] = \{\psi \in M(B) \,:\, |\psi(b_i) - \phi(b_i)| < \varepsilon,\ i = 1,\dots,n\}$$

with $\varepsilon > 0$, $b_1, \ldots, b_n \in B$ and $\phi \in M(B)$.

Theorem

$M(B)$ is a compact Hausdorff space.

Proof

$M(B)$ is Hausdorff because it is a subspace of the Hausdorff space B^*.
By Banach-Alaoglu, the unit ball is compact in the weak*-topology. We need that $M(B)$ is a closed subset of the unit ball.
Let ϕ be in the closure of $M(B)$ with respect to the unit ball such that $\phi \in B^*$ and $||\phi|| \leq 1$. To show that $\phi(ab) = \phi(a)\phi(b)$, consider

$$U_{\varepsilon;a,b,ab}[\phi] = \{\psi \in B^* \,:\, |\psi(b_i) - \phi(b_i)| < \varepsilon,\ i = 1, \ldots, n\}$$

Then $\psi \in M(B) \cap U_{\varepsilon;a,b,ab}[\phi]$, so we have

$$|\psi(a) - \phi(a)| < \varepsilon, \quad |\psi(b) - \phi(b)| < \varepsilon, \quad \text{and} \quad |\psi(ab) - \phi(ab)| < \varepsilon.$$

We know that $\psi(ab) = \psi(a)\psi(b)$. Therefore

$$\begin{aligned} |\phi(ab) - \phi(a)\phi(b)| &= |\phi(ab) - \psi(ab) - \phi(a)\phi(b) + \psi(a)\psi(b)| \\ &\leq |\phi(ab) - \psi(ab)| + |\phi(a) - \psi(a)| \cdot |\phi(b)| + |\psi(a)| \cdot |\phi(b) - \psi(b)| \\ &\leq \varepsilon + \varepsilon||b|| + \varepsilon||a|| \end{aligned}$$

Taking $\varepsilon \to 0$, $\phi(ab) = \phi(a)\phi(b)$, $\forall\, a, b \in B$. Similarly, $\phi(e) = 1$.
Thus $\phi \in M(B)$ and $M(B)$ is closed in the unit ball of B^*.

Proposition

For $b \in B$, the Gelfand transform $\hat{b} : M(B) \to \mathbb{C}$ is continuous and $||\hat{b}|| := \max_{\phi \in M(B)} |\hat{b}(\phi)| \leq ||b||$.
In other words, $\hat{b} \in C(M(B))$.

Proof

We need to show that $\hat{b}$ is continuous at each $\phi_0 \in M(B)$.
Consider $U = B_\varepsilon(\hat{B}(\phi_0))$ an ε-neighborhood in $\mathbb{C}$. Then the preimage is

$$\hat{b}^{-1}(U) = \{\phi \in M(B) \,:\, \hat{b}(\phi) \in B_\varepsilon(\hat{b}(\phi_0))\} = \{\phi \in M(B) \,:\, |\hat{b}(\phi) - \hat{b}(\phi_0)| < \varepsilon\} = \{\phi \in M(B) \,:\, |\phi(b) - \phi_0(b)| < \varepsilon = U_{\varepsilon;b}[\phi_0]$$

with $[\phi_0]$ open in $M(B)$.
Also note that

$$|\hat{b}(\phi)| = |\phi(b)| \leq ||phi|| \cdot ||b|| = ||b||$$

Corollary

The Gelfand transform of B, $\Lambda : B \to C(M(B))$ by $b \mapsto \hat{b}$ is a Banach algebra homomorphism with $||\Lambda|| = 1$

Proof

Λ is linear and multiplicative with

$$\widehat{a+b} = \hat{a} + \hat{b}$$
$$\widehat{ab} = \hat{a}\hat{b}$$

$||\Lambda|| = 1$ because $||\hat{b}|| \leq ||b||$ and $||\hat{e}|| = ||e|| = 1$.
Then $\hat{e}(\phi) = \phi(e) = 1$.
It follows also that

$$(\widehat{ab})(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi) \cdot \hat{b}(\phi) = (\hat{a} \cdot \hat{b})(\phi)$$

Corollary

For $b \in B$, $b \in GB$ if and only if $\hat{b} = \Lambda(b) \in GC(M(B))$.
As a consequence, $\mathrm{sp}_B(b) = \mathrm{sb}_{C(M(B))}\,\Lambda(b)$ (Λ preserves spectrum).

Proof

$b \in GB$ implies that $ab = e$ and that $\hat{a} \cdot \hat{b} = \hat{e} = 1$. Therefore $(\hat{b})^{-1} = \hat{a} = \widehat{b^{-1}}$.
We have also that $\hat{b} \in GC(M(B))$ implies $\hat{b}(\phi) \neq 0$, $\forall \phi \in M(B)$ and $\phi(b) \neq 0$ similarly. Therefore $b \in GB$.

Gelfand Theory

Two problems for a given Banach algebra B:

1. How to determine $M(B)$.

2. How to determine its topology.

Theorem

Let B be a commutative Banach algebra with maximal ideal space $M(B) = X$ and topology τ on X.
Now assume we have another topology ρ on X such that

1. (X, ρ) is a compact topological space.

2. $\forall\, b \in B$, $\hat{b} : X \to \mathbb{C}$ is continuous in the (X, ρ) topology.

Then $\tau = \rho$.

Proof

First show that $\tau \subseteq \rho$.
Take $U \in \tau$ from the abse of the topology

$$U = U_{\varepsilon; b_1, \ldots, b_n}[\phi] = \{\psi \in X \,:\, |\psi(b_i) - \phi(b_i)| < \varepsilon,\ \forall\, i\}$$

Then

$$U = \bigcap_{i=1}^{n} \{\psi \in X \,:\, |\hat{b}_i(\psi) - \hat{b}_i(\phi)| < \varepsilon\} = \bigcap_{i=1}^{n} (\hat{b}_i)^{-1}(B_\varepsilon(\hat{b}_i(\phi)))$$

which is open in the (X, ρ) topology because $\hat{b}_i$ is continuous. Therefore $U \in \rho$.
We have that $\mathrm{id} : (X, \rho) \to (X, \tau)$ where (X, ρ) is compact and (X, τ) is Hausdorff is continuous ($\tau \leq \rho$).
Then as an open map, we map closed sets to closed sets.
$A \subset X$ closed $\implies$ A comapct $\implies$ $\mathrm{i}(A)$ compact $\implies$ $\mathrm{id}(A)$ closed.
Therefore $(\mathrm{id})^{-1}$ is continuous and $\rho = \tau$.

Theorem

Let X be a compact Hausdorff space and $B = C(X)$.
Then $M(B)$ Is homeomorphic to X by the map

$$\tau : X \ni x \mapsto \phi_x \in M(B)$$

where $\phi_x(b) = b(x)$ is the point evaluation for $b \in B = C(X)$.

Proof

We have that ϕ_x is indeed in $M(B)$ easily.
Then to show τ is injective, $\phi_x = \phi_y$ implies that $\phi_x(b) = \phi_y(b)$, $\forall\, b \in B$. Then $b(x) = b(y)$, $\forall\, b \in C(X)$ and $x = y$.
Because $x \neq y$ implies that there exists $b \in C(X)$ such that $b(x) \neq b(y)$.
Since X is a normed space, $\{x\}$ and $\{y\}$ are closed. By Urysohn's Lemma, there exists a continuous b such that $b|_{\{x\}} = 0$ and $b|_{\{y\}} = 1$.

IMAGE 1

To see that τ is surjective, otherwise there would exist $\phi \in M(B)$ such that $\phi \neq \phi_x$, $\forall\, x \in X$.
That implies that there exists $b_x \in B$ such that $\phi(b_x) \neq \phi_x(b_x) = b_x(x)$.
Put $a_x = b_x - \phi(b_x) \cdot e$. Then $\phi(a_x) = 0$. So

$$\phi_x(a_x) = \phi_x(b_x) - \phi(b_x) \neq 0$$

IMAGE 2

With a_x continuous, we find a neighborhood $U_x \ni x$ where $a_x(t) \neq 0$, $\forall\, t \in U_x$.
For $\{U_x\}_{x \in X}$ an open cover of X, there exists a finite subcover $\{U_{x_i}\}_{i=1,\dots,N}$. Consider

$$a(t) = \sum_{i=1}^{N} |a_{x_i}(t)|^2$$

which is itself continuous and $a(t) > 0$ for each $t \in X$.
So $a \in B = C(X)$, a is invertible in B and

$$\phi(a) = \sum_{i=1}^{N} \phi(\overline{a_{x_i}}) \underbrace{\phi(a_{x_i})}_{=0}$$

which would imply that a is not invertible, a contradiction.
To show that $\tau : X \ni x \mapsto \phi_x \in M(B)$ is continuous, take $X \ni x_0 \mapsto \phi_{x_0}$ and consider

$$U = U_{\varepsilon;b_1,\ldots,b_N}[\phi_{x_0}] = \{\phi_x : |\phi_x(b_i) - \phi_{x_0}(b_i)| < \varepsilon\} = \{\phi_x \,:\, |b_i(x) - b_i(x_0)| < \varepsilon\}$$

Then

$$\tau^{-1}(U) = \{x \in X \,:\, |b_i(x) - b_i(x_0)| < \varepsilon\}$$

which is the open preimage of $b_i : X \to \mathbb{C}$ with b_i continuous.
Then $\tau : X \to M(B)$ is a continuous map between comapct, Hausdorff spaces and τ^{-1} is continuous.

Section 2.5: Commutative Banach Algebras Generated by Single Elements

Consider the Banach algebra $A \ni B$ and

$$B = \mathrm{alg}_A\{e, b\} = \mathrm{clos}_A\left\{\sum_{n=0}^{N} \lambda_n b^n\right\}$$

Theorem

The maximal ideal space $M(B)$ of $B = \mathrm{alg}_A\{e, b\}$ is homeomorphic to $\mathrm{sp}_B(b)$.

November 4, 2024

Recall

Let A be a unital Banach algebra, $b \in A$ and $B = \mathrm{alg}_A\{e, b\} = \mathrm{clos}_A\{\sum \lambda_i b^i \,:\, \lambda_i \in \mathbb{C},\, N \in \mathbb{N}$ the smallest closed subalgebra containing e and b.
Then B is a commutative Banach algebra and the closure of all polynomials in b, $p(b)$.

Theorem

The maximal ideal space of $B = \mathrm{alg}\{e, b\}$ is homeomorphic to $\mathrm{sp}_B(b)$. The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \mathrm{sp}_B(b)$$

is a homeomorphism.

Proof

- $\tau : M(B) \to \mathbb{C}$ maps into $\mathrm{sp}_B(b)$.

  Otherwise, $z = \phi(b) \notin \mathrm{sp}_B(b)$ (for some ϕ) and $b - z \cdot e$ is invertible in B. So

$$\phi(b - z \cdot e) = \phi(b) - z \cdot \phi(e) = 0$$

and, therefore,

$$\phi((b - ze)^{-1}(b - ze)) = \phi(e) = 1$$

a contradiction.

- τ is injective.

Assume that $\phi_1(b) = \phi_2(b)$.
Then $\phi_1(b^n) = \phi_2(b^n)$ for $n = 0, 1, \ldots$ and, consequently,

$$\phi_1\left(\sum_{n=0}^{N} \lambda_n b^n\right) = \phi_2\left(\sum_{n=0}^{N} \lambda_n b^n\right)$$

Because $B = \operatorname{clos}\{\sum \lambda_n b^n\}$ and ϕ_i is continuous, $\phi_1(a) = \phi_2(a)$ for each $a \in B$ and $\phi_1 = \phi_2$.

- τ is surjective.

Take $z \in \operatorname{sp}_B(b)$. Then $b - z \cdot e$ is not invertible in B.
It follows from Gelfand theorem that there exists some $\phi \in M(B)$ such that $\phi(b - ze) = 0$ and $\phi(b) = z$.

- We know $\tau : M(B) \to \operatorname{sp}_B(b)$ is an injection with the natural topology (weak*-topology of B^*) on $M(B)$ and $\operatorname{sp}_B(b)$ Hasudorff compact.

Define (another) topology on $M(B)$ via τ.
To show that both topologies are the same, we need that for each $b \in B$, $\hat{b} : M(B) \to \mathbb{C}$ is a continuous function.

$$\begin{tikzcd} \operatorname{sp}_B(b) \arrow[r, "\tau^{-1}"] \arrow[rr, bend left, "\tilde{a}"] & M(B) \arrow[r, "\hat{a}"] & \mathbb{C} \end{tikzcd}$$

equivalently, $\tilde{a} = \hat{a} \circ \tau^{-1} : \operatorname{sp}_B(b) \to \mathbb{C}$ is continuous.
Let $\phi = \tau^{-1}(z)$, $\tau(\phi) = z$ and $\hat{b}(\phi) = \phi(b) = z$.
Then if $a = e$, $\hat{e}(\phi) = \phi(e) = 1$, $\tilde{e}(z) = 1$
If $a = b$, $\hat{b}(\phi) = \phi(b) = z$ and $\tilde{b}(z) = z$ continuous.
For $a = b^n$, $\widehat{b^n} = (\hat{b})^n$ and $\widetilde{b^n} = (\tilde{b})^n$ so $\widetilde{b^n}(z) = z^n$.
When $a = p(b)$, $p(z) = \sum_{i=1}^{N} \lambda_i z^i$ a polynomial,

$$\begin{aligned}
\hat{a} &= \widehat{\sum_{i=1}^{N} \lambda_i b^i} = \sum_{i=1}^{N} \lambda_i \hat{b}_i \\
\tilde{a} &= \sum_{i=1}^{N} \lambda_i \widetilde{b^i} \\
\tilde{a}(t) &= \sum_{i=1}^{N} \lambda_i z^i = p(z)
\end{aligned}$$

For $a \in B$, $||a - a_n|| \to 0$, $a_n = p_n(b)$, we have $\hat{a} - \hat{a}_n = \widehat{a - a_n}$ and take

$$\max_{\phi \in M(B)} |\hat{a}(\phi) - \hat{a}_n(\phi)| \leq ||a - a_n||_B = \max_{z \in \operatorname{sp}_B(b)} |\tilde{a}(z) - \tilde{a}_n(z)|$$

Therefore $\tilde{a}_n \rightrightarrows \tilde{a}$ uniformly on $\operatorname{sp}_B(b)$ with $\tilde{a}$ continuous.
Both topologies on $M(B)$ coincide. Therefore $\tau : \operatorname{sp}_B(b) \to M(B)$ is a homemorphism.

Theorem

Let A be a unital Banach algebra, and $b \in A$ an invertible element. Then for

$$B = \mathrm{alg}_A\{e, b, b^{-1}\} = \mathrm{clos}_A\left\{\sum_{i=-N}^{N} \lambda_i b^i \ : \ \lambda_i \in \mathbb{C},\ N \in \mathbb{N}\right\}$$

This is the closure of all trigonometric polynomials. The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \mathrm{sp}_B(b)$$

is a homemorphism.
The proof follows similarly to that for the previous theorem.

Example: Wiener Algebra

Let $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ with $C(\mathbb{T})$ continuous functions $\mathbb{T} \to \mathbb{C}$.
$W \subseteq C(\mathbb{T})$ is the set of all functions with an absolutely convergent Fourier series.

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$$

such that $||a||_W := \sum_{n=-\infty}^{\infty} |a_n| < +\infty$.
This is a Banach algebra with addition and multiplication defined pointwise and $||ab|| \leq ||a|| \cdot ||b||$ (verify!).
Further, for $a \in W$, $||a||_{C(\mathbb{T})} \leq ||a||_W$ and $W \subseteq C(\mathbb{T})$ is a continuous embedding.

Remark

W is a Banach algebra.
W is isometrically isomorphic to $\ell^1(\mathbb{Z})$ (as a Banach algebra)

$$c(t) = a(t) \cdot b(t) \qquad \{c_n\} = \{a_n\} * \{b_n\}$$
$$\left(\sum c_n t^n\right) = \left(\sum a_n t^n\right)\left(\sum b_n t^n\right) \qquad c_n = \sum_{k \in \mathbb{Z}} a_{n-k} b_k$$

Consider $\chi_n(t) = t^n$ for $n \in \mathbb{Z}$. $\chi_0 = e$, $\chi_n = (\chi_1)^n$, $(\chi_1)^{-1} = \chi_{-1}$. Note that

$$W = \mathrm{alg}_W\{X_0, X_1, X_{-1}\}$$

Write

$$W = \mathrm{clos}_W\left\{p(\chi_1) = \sum_{i=-N}^{N} \lambda_i (\chi_1)^i\right\}$$

and, to show that the trigonometric polynomials are dense in W, for $a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$, consider $a^{(N)}(t) = \sum_{n=-N}^{N} a_n t^n$ and

$$||a(t) - a^{(N)}(t)|| = \sum_{|n|>N} |a_n|$$

which converges to 0 as $N \to +\infty$ because $\sum_{n\in\mathbb{N}} |a_n| < +\infty$.
Claim: $\mathrm{sp}_W(\chi_1) = \mathbb{T}$.
Indeed, for $|z| > 1$,

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = -\frac{1}{z}\frac{1}{1-t/z} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{t}{z}\right)^n \in W$$

For $|z| < 1$

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = \frac{1}{t}\frac{1}{1-z/t} = \frac{1}{t}\sum_{n=-\infty}^{0}\left(-\frac{t}{z}\right)^n \in W$$

However, for $|z| = 1$, $\chi_1 - z\chi_0 = t - z$ vanishes at $t = z \in \mathbb{T}$. Thus $\chi_1 - z\chi_0 \notin GW$.
Therefore, the maximal ideal space of W is homeomorphic to $\mathbb{T}$.
If we identify $M(W) \cong \mathbb{T}$ via τ from above,

$$\phi(\chi_0) = t_0 \in \mathbb{T}$$

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "\tau^{-1}"] \arrow[rr, bend right, "a"'] & M(W) \arrow[r, "\hat{a}"] & \mathbb{C}
\end{tikzcd}$$

For $a = \sum a_n t^n = \sum a_n \chi_n$,

$$\phi(a) = \sum a_n \phi(\chi_n) = \sum a_n \phi(\chi_1)^n = \sum a_n t_0^n$$

Then $\hat{a}(\phi) = \sum a_n t_0^n$ and $\hat{a}(t^{-1}(t_0)) = \sum a_n t_0^n$.
Finally $\tilde{a}(t_0) = \sum a_n t_0^n$ implies that $\tilde{a} = a$.

Theorem (Wiener)

Let $a \in W$. Then a is invertible in W if and only if $a(t) \neq 0$, $\forall\, t \in \mathbb{T}$ (i.e. if a is invertible in $C(\mathbb{T})$).

Remark

$a \in W$ and $a(t) \neq 0$, $\forall\, t \in \mathbb{T}$ implies that $\frac{1}{a} \in W$.
That is, if a has an absolutely convergent Fourier series (under these conditions) then $\frac{1}{a}$ has an absolutely convergent Fourier series.

Example

Let $A = C(\mathbb{T})$ and $B = \mathrm{alg}_A\{X_0, X_1\} = \mathrm{clos}_{C(\mathbb{T})}\left\{p(t) = \sum_{n=0}^{N} \lambda_n t^n\right\}$.
One can show that $B = C_+(\mathbb{T}) = \iota(A(\mathbb{D}))$ where

$$C_+(\mathbb{T}) = \left\{a \in C(\mathbb{T}) \,:\, \int_0^{2\pi} e^{inx} a(e^{ix})\, dx = 0,\ \forall\, n < 0\right\}$$

and

$$A(\mathbb{D}) = \{a \in C(\overline{\mathbb{D}}) \,:\, a|_{\mathbb{D}} \text{ holomorphic}\}$$
$$\iota : A(\mathbb{D}) \to a|_{\mathbb{T}} \in C(\mathbb{T})$$

Claim

$\mathrm{sp}_B(\chi_1) = \overline{\mathbb{D}}$.
For $|z| < 1$,

$$\frac{1}{\chi_1 - z} = \frac{1}{t-z} \begin{array}{l} \in C(\mathbb{T}) \\ \notin C_+(\mathbb{T}) \end{array}$$

and (for $|t| \le 1$), $\frac{1}{t-z} \notin A(\mathbb{D})$.
So $M(B) \cong \overline{\mathbb{D}}$. For $|t_0| = 1$, $\phi_{t_0}(a) = a(t_0)$ $(a \in B)$.
For $|t_0| < 1$,

$$\phi_{t_0}(a) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{a(t)}{t - t_0}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{a(e^{ix})}{1 - e^{-ix} t_0}\, dx$$

or $\phi_{t_0}(a) = a(t_0)$ (if a is holomorphically extended into $\mathbb{D}$).

November 6, 2024

Theorem

Let A be a unital Banach algebra and $b_1, \dots b_n \in A$ be commuting elements.
Then the maximal ideal space of $B = \mathrm{alg}_A\{e, b_1, \dots, b_n\}$ is homeomorphic to some compact subset of

$$K \subseteq \mathrm{sp}_B(b_1) \times \cdots \times \mathrm{sp}_B(b_n) \subseteq \mathbb{C}^n$$

That is, the map

$$\tau : M(B) \ni \phi \mapsto (\phi(b_1), \dots, \phi(b_n)) \in K \subseteq \mathbb{C}^n$$

is a homeomorphism.

Remark

$K = \mathrm{im}\, \tau = \tau(M(\phi))$.
$\phi(b_i) \subseteq \mathrm{sp}_B(b_i)$.
For $a \in B$, $\hat{A} : M(B) \to \mathbb{C}$, $\tilde{a} = \hat{a} \circ \tau^{-1} : K \to \mathbb{C}$,

$$\begin{aligned} \tilde{e}(x) &= 1 \qquad\qquad x = (x_1, \dots, x_n) \\ \tilde{b}_i(x) &= x_i \end{aligned}$$

$\Lambda : b \mapsto \hat{b} \in C(M(B))$, $\tilde{b} \in C(K)$.

Section 2.6: Shilov Idempotent and Arens-Royden Theorems

For B a unital, commutative Banach algebra,

$$\Lambda : B \to \underbrace{C(M(B))}_{M(B)}$$

Obviously, if $p \in B$ is idempotent then $(\hat{p})^2 = \hat{p}$.

Theorem: Shilov

Let $\chi \in C(M(B))$ be an idempotent.
Then there exists a unique idempotent $p \in B$ such that $\hat{p} = \chi$.

Note

Idempotents $\chi \in C(M(B))$ correspond (uniquely) to clopen (closed and open) subsets of $K \subseteq M(B)$.

$$\chi(\phi) = \begin{cases} 0 & \varnothing \notin K \\ 1 & \varnothing \in K \end{cases}$$

Theorem: Arens-Royden

The abstract index group $\kappa(B) = GB/G_0B$ is group-isomorphic to the abstract index group $\kappa(C(M(B)))$ via the map

$$\iota : \kappa(B) \ni [b] \mapsto [\hat{b}] \in \kappa(C(M(B))$$

Remark

For $\hat{B} = C(M(B))$ and $\Lambda : B \ni b \mapsto \hat{b} \in \hat{B}$,

- $GB \ni b \to \hat{b} \in G\hat{B}$.
- $G_0B \ni b \mapsto \hat{b} \in G_0\hat{B}$

$$G_0B = \{\exp(a) \,:\, a \in B\}$$
$$\exp(a) = b \mapsto \hat{b} = \exp(\hat{a}) \in G_0\hat{B}$$

- $\iota[b] \mapsto [\hat{b}]$ is well defined and

$$[b] = \{b\exp(a) \,:\, a \in B\}$$
$$[\hat{b}] = \{\hat{b}\exp(\alpha) \,:\, \alpha \in \hat{B}\}$$

- Easy: ι is a group homomorphism.
- ι is injective and surjective (non-trivial)
- ι injective means that if $b \in GB$ is such that $\hat{b}$ has a logarithm in $\hat{B}$, $\hat{b} = \exp(\alpha)$, $\alpha \in \hat{B}$, then b has a logarithm in B, $b = \exp(a)$ for some $a \in B$ (and $\hat{a} = \alpha$). Or if $b_1, b_2 \in GB$ such that $\hat{b}_1, \hat{b}_2 \in \hat{B}$ are homotopic, continuous functions $M(B) \to \mathbb{C} \setminus \{0\}$. Then b_1 and b_2 are connected by a path in GB (i.e. $b_1 = b_2\exp(a)$)
- ι surjective means that if $\gamma \in G\hat{B}$ (i.e. $\gamma : M(B) \to \mathbb{C} \setminus \{0\}$) is continuous, then there exist a $c \in GB$ such that $\hat{c} = \gamma \cdot \exp(\alpha)$ with $\alpha \in \hat{B}$.

Chapter 3: C*-Algebras

Section 3.1: Operators on Hilbert Space

- Inner-product space: a complex vector space H with inner product $H \times H \ni (x, y) \mapsto \langle x, y\rangle \in \mathbb{C}$ such that

1. $\langle x, y\rangle$ is linear in x and anti-linear in y

2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

3. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.

- Norm: $||x|| = \sqrt{\langle x, x\rangle}$
- Cauchy-Schwarz Inequality: $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$.
- Triangle Inequality: $||x + y|| \leq ||x|| + ||y||$.

Definition: Hilbert Space

A Hilbert space is a complete inner-product space.

IMAGE 1

Examples

$H = \mathbb{C}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$
$H = \ell^2(\Omega)$ (Ω possibly uncountable), $\langle x, y\rangle = \sum_{\omega\in\Omega} x_\omega \overline{y}_\omega$ where $||x||^2 = \sum_{\omega\in\Omega} |x_\omega|^2 < +\infty$
$H = L^2(S, d\mu)$ with $(S, B_s, d\mu)$ a measure space.

Theorem: Riesz Representation

For every $\phi \in H^*$, there exists a unique $y \in H$ such that $\phi(x) = \langle x, y\rangle$.
The map $\tau : H \ni y \mapsto \phi \in H^*$ is an isometric antilinear (almost) isomorphism.

$$\tau(\lambda y) = \overline{\lambda}\tau(y)$$
$$\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$$

We will consider $L(H)$ the Banach algebra of bounded linear operators $A : H \to H$ equipped with the norm

$$||A||_{L(H)} = \sup_{\substack{x\in H \\ x\neq 0}} \frac{||Ax||}{||x||}$$

Definition: Adjoint Operator (for Hilbert Spaces)

For $A \in L(H)$, its adjoint $A^* \in L(H)$ is given by $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for any $x, y \in H$.

Remark: Well-Defined

A^* is well-defined. For $y \in H$, consider $\phi(x) = \langle Ax, y\rangle$ which is a bounded, linear functional ($\phi \in H^*$).
By Cauchy-Schwarz, $|\langle Ax, y\rangle| \leq ||Ax|| \cdot ||y|| \leq ||A|| \cdot ||x|| \cdot ||y||$, so $||\phi|| \leq ||A|| \cdot ||y||$.
By the Riesz Representation Theorem, there exists $z_y \in H$ such that $\phi(x) = \langle x, z_y\rangle$.
Put $A^*(y) = z_y$, $A^* : H \ni y \mapsto z_y \in H$ such that $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

Remark: Linearity

A^* is linear.

Remark: Boundedness

A^* is bounded, $||A^*y|| = ||z_y|| = ||\phi|| \leq ||A|| \cdot ||y||$.
Therefore, $A^* \in L(H)$.

Properties

- $(A^*)^* = A$.
- $||A^*|| = ||A||$
- $(A+B)^* = A^* + B^*$ and $(\lambda A)^* = \overline{\lambda} A^*$.
- $(AB)^* = B^* A^*$
- A is invertible if and only if A^* is invertible and $(A^{-1})^* = (A^*)^{-1}$.
- $||A^*A|| = ||A||^2$.

$$
\begin{aligned}
\langle A^*Ax, x\rangle &= \langle Ax, Ax\rangle = ||Ax||^2 \\
||Ax||^2 &\leq ||A^*Ax|| \cdot ||x|| \leq ||A^*A|| \cdot ||x||^2 \\
||A||^2 &= \sup_{x\neq 0} \frac{||Ax||^2}{||x||^2} \leq ||A^*A|| \\
||A^*A|| &\leq ||A^*|| \cdot ||A|| = ||A||^2
\end{aligned}
$$

Definitions

$A \in L(H)$ is called

- Self-adjoint if $A^* = A$.
- Unitary if $A^*A = AA^* = I$
- Normal if $A^*A = AA^*$
- Positive $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, $\forall x \in H$

Positive implies self-adjoint which implies normal.
Unitary implies normal.

Proposition

$A \in L(H)$ is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$.

Proof

$A = A^*$ implies $\langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$. Therefore $\langle Ax, x\rangle \in \mathbb{R}$.
$\langle Ax, x\rangle \in \mathbb{R}$ implies $\langle Ax, x\rangle = \overline{\langle x, Ax\rangle} = \langle x, Ax\rangle = \langle A^*x, x\rangle$. Therefore $\langle (A-A^*)x, x\rangle = 0$ so (exercise) $\langle (A-A^*)x, y\rangle = 0$.
Use with $\phi(x, y) = \langle (A-A^*)x, y\rangle$, $\phi(x, y) = \frac{1}{4}(\phi(x+y, x+y) - \phi(x-y, x-y) + i\phi(x+iy, x+iy) - i\phi(x-iy, x-iy))$.

Proposition

For $A \in L(H)$, $A^* A \geq 0$.

Proof

$\langle A^* Ax, x\rangle = \langle Ax, Ax\rangle = ||Ax||^2 \geq 0$.

November 13, 2024

Recall: Propositions

A is self adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$.
$A^* A$ is positive (i.e. $\langle A^* Ax, x\rangle \geq 0$).

Theorem

Let $A \in L(H)$

1. if $A = A^*$, then $\operatorname{sp}(A) \subseteq \mathbb{R}$.

2. if $A \geq 0$, then $\operatorname{sp}(A) \subseteq [0, +\infty)$.

Lemma

If $K \in L(H)$, $||Kx|| \geq \delta||x||$, and $||K^* x|| \geq \delta||x||$ for $\delta > 0$, then K is invertible.

- Proof

  K is injective ($\ker K = \{0\}$).
  The image $\operatorname{im} K$ is closed (use cauchy sequences with $Kx_n \to y$, $\{Kx_n\}$ Cauchy, $\{x_n\}$ Cauchy, $x_n \to x$. Then $Kx_n \to Kx \implies Kx = y$).
  $\ker K^* = \{0\}$ since

$$\begin{aligned}(\ker K^*) &= (\operatorname{im} K)^\perp \\ &= \{y \in H : \langle y, z\rangle = 0,\ \forall z \in \operatorname{im} K\} \\ &= \{y \in H : \langle y, Kx\rangle = 0,\ \forall x \in H\} \\ &= \{y \in H : \langle K^* y, x\rangle = 0,\ \forall x \in H\} \\ &= \{y \in H : Ky^* = 0\}\end{aligned}$$

  and $\{0\} = (\operatorname{im} K)^\perp$ implies that $\operatorname{im} K$ is dense.
  Therefore, $\operatorname{im} K = H$, K is surjective and injective, and ultimately K is invertible.

Proof of a

Take $\lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$ ($\beta \neq 0$), and write

$$A - \lambda I = A - \alpha I - i\beta I = \beta\left(\overbrace{\frac{A - \alpha i}{\beta}}^{:=T} - i\right) = \beta(\underbrace{T - iI}_{:=K})$$

Then $K = T - iI$ and $K^* = T + iI$. So

$$||(T-iI)x||^2 \langle (T-iI)x, (T-iI)x \rangle = ||Tx||^2 + ||x||^2 - i\langle x, Tx\rangle + i\langle Tx, x\rangle$$

Since $\overline{\langle x, Tx\rangle} = \langle Tx, x\rangle$, we get $||Tx||^2 + ||x||^2 \geq ||x||^2$.
Likewise, $||(T+iI)x||^2 \geq ||x||^2$ so $T - iI$ is invertible and $A - \lambda I$ is invertible.

Proof of b

Recall that positive implies self-adjoint.
For $\lambda \in (-\infty, 0)$, $\lambda = -\alpha$, $\alpha > 0$, we have $A - \lambda I = A + \alpha I =: K$. Then

$$||Kx||^2 = \langle (A+\alpha I)x, (A+\alpha I)x\rangle = ||Ax||^2 + \alpha^2 ||x||^2 + \alpha(\langle Ax, x\rangle + \langle x, Ax\rangle)$$

By assumption, $\langle Ax, x\rangle \geq 0$ and $\langle x, Ax\rangle \geq 0$, so $||Kx||^2 \geq \alpha^2 ||x||^2$.

Remark

For $A \in L(H)$ where $A = A^*$, the following are equivalent

1. A is positive.
2. $\mathrm{sp}(A) \subseteq [0, +\infty)$.
3. $A = B^* B$ for some $B \in L(H)$.

Section 3.2: C* Algebras

Definition:

A C^*-algebra is a Banach algebra B which has a map

$$* : B \ni a \mapsto a^* \in B$$

(called an involution) such that

- $(a^*)^* = a$, $(ab)^* = b^* a^*$, and $e^* = e$.
- $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \overline{\lambda} a^*$.
- $||a^* a|| = ||a||^2$.

Remark

If B has a unit, $e^* = e$.
For invertible elements, b invertible if and only if b^* is invertible and $(b^*)^{-1} = (b^{-1})^*$.
$||a^*|| = ||a||$. Indeed $||a||^2 = ||a^* a|| \leq ||a|| \cdot ||a^*||$ so $||a|| \leq ||a^*||$ (for $a \neq 0$).
Since $a \mapsto a^*$, $||a^*|| \leq ||a^{**}|| = ||a||$.

Examples

- $B = L(H)$ bounded linear operators on Hilbert spaces.
- $B = C(X)$ with X a compact Hausdorff space given by $a : X \to \mathbb{C}$ continuous and $a^*(x) := \overline{a(x)}$ (complex conjugate).
- $B = L^\infty(S, d\mu)$ essentially bounded functions on a measure space (S, B_S, μ) again with $a^*(x) := \overline{a(x)}$.

Non-examples

$B = W = \{\sum_{n\in\mathbb{Z}} a_n t^n \,:\, \sum |a_n| < +\infty\}$. We have $||a|| = \sum |a_n|$ and $||a^*|| = ||a||$ but not $||a^* a|| = ||a||^2$.
$B = C^1[0,1]$.

Definitions:

An element $b \in B$ is called

- self adjoint if $b^* = b$
- unitary if $b^* b = bb^* = e$
- normal if $b^* b = bb^*$
- positive if $b^* = b$ and $\mathrm{sp}(b) \subseteq [0, +\infty]$.

Proposition

For $b \in B$ normal, the spectral radius $r(b) := \max\{|\lambda| \,:\, \lambda \in \mathrm{sp}(b)\} = ||b||$.

Proof

We know that $r(b) = \lim_{n\to\infty} ||b^n||^{1/n} = \inf_{n\in\mathbb{N}} ||b^n||^{1/n}$. Therefore, $r(b) \le ||b||$.
For any element a which is self adjoint, we have $||a^* a|| = ||a^2|| = ||a||^2$. By induction, $||a^{2^k}|| = ||a||^{2^k}$ so $||a^n||^{1/n} = ||a||$ for $n = 2^k$.
For a normal element b,

$$||b^n||^2 = ||(b^n)^* b^n|| = ||(b^* b)^n||$$

Then, since $b^* b$ is self-adjoint, $||b^n||^2 = ||b^* b||^n$ for $n = 2^k$ and $||b^n||^{1/n} = ||b^* b||^{1/2}$.
Therefore $r(b) = ||b^* b||^{1/2} = ||b||$.

Corollary

The norm in a C^*-algebra is uniquely determined (i.e. there are no different equivalent norms).

Proof

Let $a \in B$ be arbitrary. Then $r(a^* a) = ||a^* a|| = ||a||^2$.
Therefore $||a|| = \sqrt{r(a^* a)}$. The spectral radius (and the spectrum) are determined in terms of algebraic properties.

Proposition

The spectrum of

1. a unitary element is contained in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

2. a self-adjoint element is contained in $\mathbb{R}$ (in fact, it is a subset of $[-||a||, ||a||]$).

Proof of 1

For a unitary element, $||a||^2 = ||a^* a|| = ||e|| = 1$, so $||a|| = 1$ and $\mathrm{sp}(a) \subseteq \mathbb{T}$.
Since $a^{-1} = a^*$, $(a^{-1})^* a^{-1} = a^{-1}(a^{-1})^* = e$, a^{-1} is also unitary and $\mathrm{sp}(a^{-1}) \subseteq \mathbb{T}$.
Observe that $\mathrm{sp}(a) = \left\{\frac{1}{\lambda} : \lambda \in \mathrm{sp}(a^{-1})\right\}$, then $\mathrm{sp}(a) \subseteq \mathbb{T}$ and $\mathrm{sp}(a^{-1}) \subseteq \mathbb{T}$.

Proof of 2

Recall that $\exp(c) = \sum_{n=0}^{\infty} \frac{c^n}{n!}$. Then $(\exp(c))^* = \exp(c^*)$.
Let a be a self adjoint element, then $u = \exp(ia)$ is unitary since $u^* = \exp((ia)^*) = \exp(-ia) = u^{-1}$.
By the spectral mapping theorem,

$$\exp(i\,\mathrm{sp}(a)) = \exp(\mathrm{sp}(ia)) = \mathrm{sp}(\exp(ia)) \subseteq \mathbb{T}$$

Therefore $\mathrm{sp}(a) \subseteq \mathbb{R}$.

Proposition

Each $b \in B$ can be written uniquely as $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint.

Proof

Define $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2i}$ which are self-adjoint and compute

$$b_1 + ib_2 = b \quad \text{and} \quad b_1 - ib_2 = b^*$$

For uniqueness, assume that $0 = b_1 + ib_2$ with b_1 and b_2 self-adjoint. Then $b_1 = -ib_2$ and, since $\mathrm{sp}(b_1) \subseteq \mathbb{R}$ and $\mathrm{sp}(b_2) \subseteq \mathbb{R}$. Then $\mathrm{sp}(b_1) = \mathrm{sp}(b_2) = \{0\}$. Therefore $||b_1|| = r(b_1) = 0$, and, since b_1 is self-adjoint, $b_1 = 0$ (similarly for b_2).

Theorem

Let $B \subseteq A$ be a unital, C^*-subalgebra of a C^* subalgebra A. Then B is inverse closed in A (i.e. $GA \cap B = GB$).

Remarks

B is inverse-closed if and only if $\forall b \in B$, if b invertible in A then $b^{-1} \in B$.
Equivalently, $\mathrm{sp}_A(b) = \mathrm{sp}_B(b)$ for every $b \in B$.

Proof

Let $b \in B$ be invertible in A. Then $c = b^* b$ is invertible in A ($c^{-1} = b^{-1}(b^{-1})^* \in A$.
Now c is a self-adjoint, therefore $\mathrm{sp}_A(c) \subseteq [-||c||, ||c||] \subseteq \mathbb{R}$.

For subalgebras, $\mathrm{sp}_B(c) \supseteq \mathrm{sp}_A(c)$ and $\partial\,\mathrm{sp}_B(c) \subseteq \partial\,\mathrm{sp}_A(c)$. In fact, $\mathrm{sp}_B(c) = \mathrm{sp}_A(c) \cup \bigcup_{\omega\in\Omega} H_\omega$ where H_ω are (bounded) connected components of $\mathbb{C} \setminus \mathrm{sp}_A(c)$.
However, because $\mathrm{sp}_A(c) \subseteq \mathbb{R}$ there are no holes. Therefore $\mathrm{sp}_B(c) = \mathrm{sp}_A(c)$.
We know that $0 \notin \mathrm{sp}_A(c)$ so $0 \notin \mathrm{sp}_B(c)$. So c is invertible in B, so b is left invertible in B since

$$e = c^{-1}c = \underbrace{c^{-1}b^*}_{\in B} b$$

Similarly, b is right invertible by repeating the argument with $d = bb^*$. Therefore b is invertible in B.

November 18, 2024

Section 3.3: Commutative C*-Algebras

Proposition

For a unital commutative C^*-algebra B and $\phi \in M(B)$, we have $\phi(b^*) = \overline{\phi(b)}$, $\forall\, b \in B$.

Proof

Write $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint. Then $b^* = b_1 - ib_2$.
Then $\mathrm{sp}(b_i) \subseteq \mathbb{R}$ implies that $\phi(b_i) \in \mathbb{R}$, otherwise $\phi(b_i) = z \in \mathbb{C} \setminus \mathbb{R}$ gives $\phi(b_1 - ze) = 0$ which implies $b_1 - ze$ is not invertible and $z \in \mathrm{sp}(b_i)$.
Then $\phi(b) = \phi(b_1) + i\phi(b_2)$ and $\phi(b^*) = \phi(b_1) - i\phi(b_2)$ with $\phi(b_i)$ real.

Remarks

A map $\phi : B \to \mathbb{C}$ is a $*$-homomoprhism if $\phi(b^*) = \phi(b)^* = \overline{\phi(b)}$ (involution in $\mathbb{C}$).
The Gelfand transform is also a $*$-homomorphism

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

with $\Lambda(b^*) = (\Lambda(b))^*$ and $\widehat{b^*} = (\hat{b})^*$. So

$$\widehat{b^*}(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{\hat{b}(\phi)} = (\hat{b})^*(\phi)$$

Theorem: Stone-Weierstrass

Let X be a compact Hausdorff space and $\mathfrak{a} \subseteq C(X)$ a subalgebra of $C(X)$ such that

- $\mathfrak{a}$ is norm-closed in $C(X)$
- $I \in \mathfrak{a}$
- if $f \in \mathfrak{a}$ then $\overline{f} \in \mathfrak{a}$
- $\forall\, x, y \in X$ with $x \neq y$, there exists $f \in \mathfrak{a}$ such that $f(x) \neq f(y)$.

$C_r(X)$ the set of all continuous functions $f : X \to \mathbb{R}$ and $\mathfrak{a}_r = \mathfrak{a} \cap C_r(X)$.

Proof

It is enough to show that $\mathfrak{a}_r = c_R(X)$ for $f \in C(X)$, $f = \operatorname{Re} f + i \operatorname{Im} f$ then $f \in \mathfrak{a}$.
If $f \in \mathfrak{a}_r$, then $|f| \in \mathfrak{a}_r$. Without loss of generality, $||f|| \le 1$, $|f(x)| \le 1$. Note that

$$\phi(t) = \sqrt{1-t} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^n$$

which converges uniformly on $[-1,1]$. Write $|f| = \sqrt{|f|^2} = \sqrt{f^2} = \sqrt{1-(1-f^2)}$. Then for $t = 1 - f^2(x) \in [0,1]$,

$$|f(x)| = \left| \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (1 - f^2(x))^n \right| = \sum_{n=0}^{N} (\cdots) + R_N(1 - f^2(x))$$

where the first term is polynomial in $f^2 = \overline{f} f \in \mathfrak{a}_r$ and we may estimate the remainder

$$\sup_{x \in X} |R_n(\cdots)| \le \sup_{t \in [0,1]} |R_N(t)$$

which converges to 0 as $N \to \infty$. Therefore $|f|$ may be approximated by elements in $\mathfrak{a}_r$.
Then $\mathfrak{a}_r$ is a lattice. Given $f, g \in \mathfrak{a}_r$, we have

$$f \vee g = \max\{f,g\} = \frac{1}{2}\{f + g + |f-g|\} \quad \text{and} \quad f \wedge g = \min\{f,g\} = \frac{1}{2}\{f + g - |f-g|\}$$

and $f \vee g, f \wedge g \in \mathfrak{a}_r$.
Now, $\forall \alpha, \beta \in \mathbb{R}$, there exists $f \in \mathfrak{a}_r$ such that $f(x) = \alpha$ and $f(y) = \beta$.
First, we obtain a function $h \in \mathfrak{a}$ with $h(x) \ne h(y)$ and $1 \in \mathfrak{a}$, then put $f = \gamma 1 + \delta h$. We need

$$\alpha = \gamma + \delta h(x) \quad \text{and} \quad \beta = \gamma + \delta h(y)$$

which can be found by solving $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & h(x) \\ 1 & h(y) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$.
Now we replace f by $\operatorname{Re} f = \frac{1}{2}(f + \overline{f}) \in \mathfrak{a}$ so $\operatorname{Re} f \in \mathfrak{a}_r$.
Let $f \in C_r(X)$. We want to show that f can be approximated by functions in $\mathfrak{a}_r$.
Let $\varepsilon > 0$, fix $x_0 \in X$ and take $x \in X$ arbitrary. Find $g_x \in \mathfrak{a}_r$ such that $g_x(x_0) = f(x_0)$ and $g_x(x) = f(x)$.
Since these functions are continuous, we have that $g_x(y) \le f(y) + \varepsilon$ in some open neighborhood $U_x \ni x$. Then U_x covers X as we range across $x \in X$. We have a finite subcover $U_{x_1}, \ldots, U_{x_N}$ and $h_{x_0} = g_{x_1} \wedge g_{x_2} \wedge \cdots \wedge g_{x_N}$.
Then $h_{x_0}(y) \le f(y) + \varepsilon$ for every $y \in X$ where $h_{x_0}(x_0) = f(x_0)$.
Doing this for all $x_0 \in X$ gives a collection $\{h_{x_0}\}_{x_0 \in X}$. For some $V_{x_0} \ni x_0$, $h_{x_0}(y) \ge f(y) - \varepsilon$.
Then $\{V_{x_0}\}_{x_0 \in X}$ covers X and admits a finite subcover $V_{x_1}, \ldots, V_{x_n}$ which gives rise to $h = h_{x_1} \vee h_{x_2} \vee \cdots \vee h_{x_n}$.
Therefore $f(y) - \varepsilon \le h(y) \le f(y) + \varepsilon$. Therefore $|f(y) - h(y)| \le \varepsilon$ for every $y \in X$ and $h \in \mathfrak{a}_r$ (by the lattice property).

Theorem

Let B be a unital C^*-algebra.
Then the Gelfand transform $\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$ is an isometric $*$-isomorphism.

Proof

We know that Λ is a Banach algebra homomorphism (i.e. multiplicative, linear and $||\Lambda|| = 1$).
We know also that Λ is a $*$-homomorphism with $\Lambda(b^*) = (\Lambda(b))^*$.

The Gelfand transform also preserves spectrum where $\mathrm{sp}_B(b) = \mathrm{sp}_{C(M(B))}(\Lambda(b))$ because $b \in B$ is invertible if and only if $\phi(b) = \hat{b}(\phi)$ is non-zero for each $\phi \in M(B)$ and, equivalently, $\hat{b} = \Lambda(b)$ is invertible in $C(M(B))$. Therefore the spectral radius is the same $r(b) = r(\Lambda(b))$.
Take $b = a^*a$ for arbitrary $a \in B$. Then $||a^*a|| = ||\Lambda(a^*a)|| = ||\Lambda(a)^*\Lambda(a)||$, so $||a||^2 = ||\Lambda(a)||^2$ and $||a| = ||\Lambda(a)||$ an isometry. It follows that Λ is injective ($\Lambda(a) = 0 \implies a = 0$).
To show that Λ is surjective ($\mathrm{im}\,\Lambda = C(M(B))$), put $\mathfrak{a} = \mathrm{im}\,\Lambda$. Then

- $\mathfrak{a}$ is an algebra
- $\mathfrak{a}$ is isometrically isomorphic to B ($\Lambda : B \to \mathfrak{a} \subseteq C(M(B))$)
- $\mathfrak{a}$ is norm closed (via Cauchy sequences)
- $1 \in \mathfrak{a}$
- $f \in \mathfrak{a}$ implies $\overline{f} \in \mathfrak{a}$ since $f = \Lambda(b)$ gives $\overline{f} = \Lambda(b)^* = \Lambda(b^*)$.
- for $\phi_1, \phi_2 \in M(B)$, $\phi_1 \neq \phi_2$ implies there exists $b \in B$ such that $\phi_1(b) \neq \phi_2(b)$, so $\hat{b}(\phi_1) \neq \hat{b}(\phi_2)$.

Therefore, we may apply Stone-Weierstrass to see that $\mathfrak{a} = C(M(B))$ implies Λ is surjective.

Examples

- $B = \ell^\infty(\mathbb{N}) = \{\{x_n\}_{n=0}^\infty : ||x|| = \sup_n |x_n| < +\infty\}$ is a commutative C^*-algebra. Therefore ℓ^∞ is isometrically $*$-isomorphic to $C(M(\ell^\infty)$. Some maximal ideals can be identified with $\mathbb{N}$.

$$\phi_N : x = \{x_n\}_{n=0}^\infty \mapsto X_N$$
$$\mathbb{N} \cong \{\phi_N\}_{N=0}^\infty \subseteq M(\ell^\infty(\mathbb{N})) = X$$

where ϕ_N are dense in X. All other functionals are given by axiom of choice. X may be identified with the ech-compcatification of $\mathbb{N}$ which is uncountable.

- $B = L^\infty([0,1])$. $X = M(B)$ is an exotic space. It is "totally disconnected" and uncountable, but $B \cong C(X)$.
- $B = PC(\mathbb{T})$ gives a maximal ideal space that can be roughly identified as $X \cong \mathbb{T} \times \{0,1\}$ with a non-trivial topology.
- Finitely generated algebras. Caution: if $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$, $B = \mathrm{alg}\{e, a\}$ may not be a C^*-algebra since a^* may not be in B. If $B = \mathrm{alg}\{e, a, a^*\}$, then it is a C^*-algebra but may not be commutative.
- Take $\mathcal{A}$ a C^*-algebra, $a \in A$ with $B = \mathrm{alg}\{e, a\}$ with $a = a^*$ or $B = \mathrm{alg}\{e, a, a^*\}$ with $aa^* = a^*a$. Then these are both unital, commutative C^*-algebras. In these cases, $B \cong C(X)$ where $X \cong \mathrm{sp}_{\mathcal{A}}(a)\,\mathrm{sp}_B(a)$. We have a homeomorphism

$$\tau : M(B) \ni \phi \mapsto \phi(a) \in \mathrm{sp}(a)$$

with $\phi(a^*) = \overline{\phi(a)}$.

November 22, 2024

Recall

Given a unital, commutative C^*-algebra B, $B \cong C(X)$ where $X = M(B)$.

Proposition

Let B be a C^*-algebra such that $B = \operatorname{alg}\{e, b, b^*\}$ where $b^* b = bb^*$ (i.e. b is normal).
Then B is commutative and $M(B)$ is homeomorphic to $X = \operatorname{sp} b$ by

$$\tau : M(B) \to \phi(b) \in X = \operatorname{sp} b.$$

In particular, B is isometrically $*$-isomorphic to $C(x)$ by

$$\tilde{\Lambda} : B \ni a \mapsto \hat{a} \circ \tau^{-1} = \tilde{a} \in C(X)$$

Remarks

- $\operatorname{sp} b^* = \overline{\operatorname{sp} b}$; $\phi(b^*) = \overline{\phi(b)}$.
- $a \in B \xrightarrow{\Lambda} \hat{a} \in C(M(B)) \to \tilde{a} = (\hat{A} \circ \tau^{-1}) \in C(X)$
- $\tilde{b}(x) = \hat{b}(\tau^{-1}(x)) = \hat{b}(\phi) = \phi(b) = x$.
- $b \mapsto \tilde{b}(x) = x$, $\tilde{b} \in C(\operatorname{sp} b)$.
- $\tilde{b}^*(x) = \widehat{b^*}(\tau^{-1}(x)) = \hat{B}^*(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{x}$.

Section 3.3.5: Functional Calculus for Normal Elements in C*-Algebras

For a continuous function $g \in C(X)$ where $X = \operatorname{sp}(b)$, $b \in B$ normal (i.e. $b^* b = bb^*$, define $g(b)$ such that $\widetilde{g(b)}(x) = g(x)$ for $g(b) \in \operatorname{alg}\{e, b, b^*\}$ (i.e. $g(b) = \tilde{\Lambda}^{-1}(g)$ where $\tilde{\Lambda} : B = \operatorname{alg}\{e, b, b^*\} \to C(\operatorname{sp} b)$).
Then $b \mapsto \hat{b}(x) = x$, $b^* \mapsto \widehat{b^*} = \overline{x}$ and $g(b) \mapsto g(x)$.
In particular, if $p(x) = \sum p_i x^i$, then $p(b) = \sum p_i b^i$.
$q(x) = \sum_{i,j} p_{ij} x^i \overline{x}^j$ and $q(b) = \sum_{i,h} p_{i,j} b^i (b^*)^j$.

Spectral Mapping Theorem

If b is normal and $g \in C(\operatorname{sp} b)$, then $\operatorname{sp}_B(g(b)) = g(\operatorname{sp}_B b)$ ($= \operatorname{im} g$).

Section 3.4: Positive Elements in C*-Algebras

Recall

$a \geq 0$ (a positive) if and only if $a^* = a$ and $\operatorname{sp} a \subseteq [0, +\infty)$.
This generalizes to $A \in L(H)$ where $A \geq 0$ if and only if $\forall x \in H$, $\langle Ax, x \rangle \geq 0$.
For $a \in C(X)$, $a \geq 0$ if and only if $a(x) \geq 0$, $\forall x \in X$.

Proposition

If $a \geq 0$ and $b \geq 0$, then $a+b \geq 0$.

Proof

If $a \geq 0$ and $b \geq 0$, then both a and b are self-adjoint and $(a+b)^* = (a+b)$.
Then $\operatorname{sp} a \subseteq [0, ||a||]$ and $\operatorname{sp} b \subseteq [0, ||b||]$ implies that $\operatorname{sp}\left(a - \frac{||a||}{2}e\right) \subseteq \left[-\frac{||a||}{2}, \frac{||a||}{2}\right]$ and $\operatorname{sp}\left(b - \frac{||b||}{2}e\right) \subseteq \left[-\frac{||b||}{2}, \frac{||b||}{2}\right]$.
Since they are self adjoint,

$$\left|\left|a - \frac{||a||}{2}e\right|\right| = r\left(a - \frac{||a||}{2}e\right) \leq \frac{||a||}{2} \quad \text{and} \quad \left|\left|b - \frac{||b||}{2}e\right|\right| = r\left(b - \frac{||b||}{2}e\right) \leq \frac{||b||}{2}$$

So

$$\left|\left|a + b - \frac{||a|| + ||b||}{2}e\right|\right| \leq \left|\left|a - \frac{||a||}{2}e\right|\right| + \left|\left|b - \frac{||b||}{2}e\right|\right| \leq \frac{||a|| + ||b||}{2}$$

Therefore $r\left(a + b - \frac{||a||+||b||}{2}e\right) \leq \frac{||a||+||b||}{2}$ and

$$\operatorname{sp}\left(a + b - \frac{||a|| + ||b||}{2}e\right) \subseteq \left[-\frac{||a|| + ||b||}{2}, \frac{||a|| + ||b||}{2}\right]$$

We conclude that $\operatorname{sp}(a+b) \subseteq \left[0, \frac{||a||+||b||}{2}\right] \subseteq [0, +\infty)$.

Theorem: Square Roots of Positive Operators

Let $a \geq 0$. Then there exists a unique element $b \geq 0$ such that $a = b^2$ (notation: $b = \sqrt{a}$).

Proof: Existence

(Using functional calculus)
Consider $B = \operatorname{alg}\{e, a\}$ with $a = a^*$ a C^*-algebra. Then $B \cong C(X)$ for $X = \operatorname{sp} a \subseteq [0, +\infty)$.
$\tilde{a}(x)$ is continuous and positive, so $\sqrt{\tilde{a}(x)}$ is also continuous and positive and therefore an element of $C(X)$.
Then there must exist $b \in B$ such that $\tilde{b}(x) = \sqrt{\tilde{a}(x)}$. So $(\tilde{b})^2 = a$, $\tilde{b}(x) \geq 0$ and $b^2 = a$ with $b \geq 0$.

Proof: Uniqueness

Assume $a = b^2 = c^2$ where $b = \sqrt{a}$ as above and $c \geq 0$. We have $b \in \operatorname{alg}\{e, a\}$.
Obviously, $ca = ac$ implies $ca^n = a^n c$ and $cx = xc$ for all $x \in \operatorname{alg}\{e, a\}$. So $cb = bc$.
Now consider $B_0 = \operatorname{alg}\{e, a, c\}$, $a^* = a$, $c^* = c$ and $ac = ca$ a commutative C^*-algebra. Then $b = \sqrt{a} \in B_0$, $a = b^2 = c^2$, $b \geq 0$ and $c \geq 0$ implies that $\hat{a} = (\hat{b})^2 = (\hat{c})^2$, $\hat{b} \geq 0$, $\hat{c} \geq 0$ (continuous functions on $M(\cdots)$). Therefore $\hat{b} = \hat{c}$ and $b = c$.

Lemma

For $a = a^*$, there exist $a_+, a_- \geq 0$ such that $a = a_+ - a_-$, $a_+ a_- = a_- a_+ = 0$.

Proof

For $B = \operatorname{alg}\{e, a\}$ a commutitve C^*-algebra, we apply Gelfand Theory / Functional Calculus. For $x \in M(B)$, define

$$\hat{a}_+ = \begin{cases} \hat{a}(x) & \hat{a}(x) \geq 0 \\ 0 & \hat{a}(x) < 0 \end{cases} \quad \text{and} \quad \hat{a}_- = \begin{cases} -\hat{a}(x) & \hat{a}(x) \leq 0 \\ 0 & \hat{a}(x) > 0 \end{cases}$$

So $a_+, a_- \in B$ and $\hat{a}_\pm C(M(B))$.

Theorem:

Let $b \in B$ be an arbitrary element in a unital C^*-algebra. Then $b^* b \geq 0$.

Remarks

- Obviously true for $B = L(H)$ or $B = C(X)$.
- It was open for quite some time for general C^*-algebras. It would be trivial if every C^*-algebra were isomorphic to some $*$-subalgebra of $L(H)$. This turns out to be true.
- Recall that $b^* b \geq 0$ boils down to whether $e + b^* b$ is invertible.

Proof

$b^* b$ is self-adjoint. Therefore $b^* b = c - d$ for $c \geq 0$ and $d \geq 0$ with $cd = dc = 0$.
To show: $d = 0$. Consider $(bd)^*(bd) = d(b^* b)d = -d^3 \leq 0$ (i.e. $d^3 \geq 0$).
Then take $bd = s + it$ where s and t are self-adjoint. Since s and t are self-adjoint, $s^2 \geq 0$, $t^2 \geq 0$ and $(s^2 + t^2) \geq 0$. So $(bd)^*(bd) + (bd)(bd)^* = 2(s^2 + t^2) \geq 0$.
Therefore $(bd)(bd)^* = 2(s^2 + t^2) + d^3 \geq 0$. We know that $(bd)(bd)^*$ and $(bd)^*(bd)$ have the same spectrum (except possibly $\{0\}$). Recall from the homework that for $\lambda \neq 0$, $\lambda - xy$ is invertible if and only if $\lambda - yx$ is invertible.
Therefore the spectrum of $(bd)(bd)^*$ and $(bd)^*(bd)$ is $\{0\}$ and

$$||bd||^2 = ||(bd)^*(bd)|| = r((bd)^*(bd)) = 0$$

so $bd = 0$ implies $d^3 = 0$, $0 = r(d^3) = (r(d))^3 = ||d||^3$ and therefore $d = 0$.

Corollary

An element $a \in B$ is positive ($a \geq 0$) if and only if $a = b^* b$ for some $b \in B$.

Proof

($\Longleftarrow$) by previous theorem.
($\Longrightarrow$) $a \geq 0$ (a self-adjoint) means $b = \sqrt{a}$. Therefore $b \geq 0$ and $b^2 = a$ which implies $b = b^*$ and $a = b^* b$.

Theorem: Polar Decomposition

Let $a \in B$ be invertible. Then there exists a unique unitary element $u \in B$ and positive element $r \in B$ ($r \geq 0$) such that $a = u \cdot r$.

Proof: Existence

Define $r = \sqrt{a^* a}$. Since a and a^* are invertible, $\mathrm{sp}(a^* a) \subseteq [\delta, +\infty)$ and r is also invertible.
Put $u = a \cdot r^{-1}$ such that $u^* u = r^{-*} a a^* r^{-1} = r^{-1} r^2 r^{-1} = e$ and $uu^* = ar^{-1} a^{-*} a^* = ar^{-2} a^* = a(a^* a)^{-1} a = e$.

Proof: Uniqueness

Write $a = u_1 r_1 = u_2 r_2$ and consider $a^* a = (r_1)^2 = (r_2)^2$ for $r_1, r_2 \geq 0$ which implies $r_1 = r_2$. It follows that $u_i = a r_i^{-1}$ is likewise unique.

Remarks

- left / right polar decompositions $a = ur = sv$ for $r, s \geq 0$ and u, v unitary.
- for $A \in L(H)$ (not in general, not even for $C(X)$) (but it does work for $L^p(S)$), $A = U \cdot R$ for $R \geq 0$ and U a partial isometry ($UU^* U = U$ or equivalently that $(U^* U)^2 = U^* U$). $\ker A$ and $\ker A^*$ may not be trivial.

Corollary

If two Hilbert spaces are isomorphic as Banach spaces, then they are isomorphic as Hilbert spaces.

Proof

Take H_1 and H_2 Hilbert spaces with $A : H_1 \to H_2$ an invertible bounded linear operator.
Define $R = (A^* A)^{1/2} : H_1 \to H_1$. Then $A = U \cdot R$ for $U : H_1 \to H_2$ unitary gives $U^* U = I_{H_1}$ and $UU^* = I_{H_2}$. So

$$H_1 \overset{R}{\to} H_1 \overset{U}{\to} H_2$$

where U is a Hilbert space isomorphism and isometry. $U^* U = I_{H_1}$ gives

$$\langle x, y \rangle_{H_1} = \left\langle U^* U x, y \right\rangle_{H_1} = \langle Ux, Uy \rangle_{H_2}$$

so U preserves $\langle \cdot, \cdot \rangle$ and $||x|| = ||Ux||$.

November 25, 2024

Section 3.5: -ideals, *-homomorphisms and Quotients in C-algebras

Definition: *-homomorphism

A homomorphism $\phi : A \to B$ such that $\phi(a^*) = \phi(a)^*$.

Definition: *-ideal

An ideal $J \subseteq A$ such that $a \in J$ implies that $a^* \in J$.

Definition: *-subalgebra

$A \subseteq B$ is a subalgebra if it is an algebra and $a \in A$ implies $a^* \in A$.

Proposition

If $\phi : A \to B$ is a $*$-homomorphism between C^*-algebras, then $\ker \phi$ is a $*$-ideal and $\operatorname{im} \phi$ is a $*$-subalgebra.

Theorem

Let $I \subseteq A$ be a closed, two-sided ideal in a $C*$-algebra. Then I is a $*$-ideal.

Proof

Take $a \in I$, consider $b = a^* a \in I$. Consider $u_n(t) = \frac{t}{1/n + t}$ for $t \geq 0$.

IMAGE 1

Then $u_n(b) = b \cdot \left(\frac{1}{n}e + b\right)^{-1} \in I$.

IMAGE 2

For $0 \le u_n(t) \le 1$,

$$0 \le (u_n(t) - 1)^2 \cdot t = \left(\frac{\frac{1}{n}}{\frac{1}{n} + t}\right)^2 \cdot t \le \frac{1}{4n}$$

With $u_n \in C([0, ||b||])$ and $\operatorname{sp} b \subseteq [0, ||b||]$, $||(u_n(t) - 1)^2 t||_{C([0,||b||])} \to 0$ and, equivalently, $||(u_n(b) - e)^2 b||_A \to 0$.
Since $(u_n(b) - e)^* = u_n(b) - e$, it follows that

$$\begin{aligned} ||(u_n(t) - 1)^2 t||_{C([0,||b||])} &= ||(u_n(b) - e)^2 b||_A \\ &= ||(u_n(b) - e)b(u_n(b) - e)|| \\ &= ||(u_n(b) - e)^* a^* a(u_n(b) - e)|| \\ &= ||a(u_n(b) - e)||^2 \\ &= ||(u_n(b) - e)a^*||^2 \to 0 \end{aligned}$$

Since $u_n(b) \in I$, $u_n(b)a^* \in I$ and since I is closed and $u_n(b)a^* \to a^*$, $a^* \in I$.

Theorem

Let A be a C^*-algebra and $I \subseteq A$ be a closed $*$-ideal. Then A/I is a C^*-algebra.

Proof

A/I is a Banach algebra with operations $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$ where $[a] = a + I = \{a + i \,:\, i \in I\}$.
We may further define $[a]^* = [a^*]$ and a norm $||[a]|| = \inf_{i \in I} ||a + i||_A$.
We want to show that $||[a]||^2 = ||[a]^*[a]||^2$.

$$||[a]_I|| = \inf_{i \in I} ||a + i|| = \inf_{i \in I} ||a^* + i^*|| = \inf_{i \in I} ||a^* + i|| = ||[a^*]|| = ||[a]^*||$$

By the sub-multiplicativity of the norm, $||[a]^*[a]|| \le ||[a^*]|| \cdot ||[a]|| \le ||[a]||^2$. In the other direction, write $0 \le z \le e$ to mean $z \ge 0$ and $e - z \ge 0$ (i.e. $z = z^*$ and $\operatorname{sp} z \subseteq [0, 1]$). Then for $z \in I$,

$$||[a]|| = \inf_{i \in I} ||a + i|| \ge \inf_{0 \le z \le e} || \ge ||[a]||$$

Since $||e - z|| \le 1$,

$$\begin{aligned} ||[a]||^2 &= \inf\left\{||a - az||^2 \,:\, 0 \le z \le e,\, z \in I\right\} \\ &= \inf\left\{||(e - z)a^* a(e - z)|| \,:\, 0 \le z \le e,\, z \in I\right\} \\ &\le \left\{||a^* a(e - z)|| \,:\, 0 \le z \le e,\, z \in I\right\} \\ &= ||[a^* a]|| \end{aligned}$$

Theorem

Let A, B be C^*-algebras and $\phi : A \to B$ a $*$-homomorphism. Then $\operatorname{im}\phi = \phi(A)$ is a closed $*$-subalgebra of B. If ϕ is injective, then it is an isometric $*$-isomorphism to its image.

Proof

Assume ϕ is injective.
Obviously, $\operatorname{sp}(\phi(a)) \subseteq \operatorname{sp} a$ since $a - \lambda e$ invertible implies $\phi(a) - \lambda e$ is invertible.
Claim: for $a = a^*$, $\operatorname{sp}(\phi(a)) = \operatorname{sp}(a)$. Otherwise, there would exist $\lambda_0 \in \operatorname{sp}(a)$ with $\lambda_0 \notin \operatorname{sp}(\phi(a))$.

IMAGE 3

So there must exist $f : \operatorname{sp}(a) \to [0,1]$ continuous with $f|_{\operatorname{sp}(\phi(a))} = 0$ and $f(\lambda_0) = 1$.

IMAGE 4

Therefore $\operatorname{sp} f(\phi(a)) = f(\operatorname{sp}(\phi(a))) = 0$ and, since f is real valued and $\phi(a)$ is self adjoint $f(\phi(a))$ suffices. It follows that $f(\phi(a)) = 0 = \phi(f(a))$ and that $f(a) = 0$. However, this contradicts our construction of f which set $f(\lambda_0) = 1$.
Hence, $\operatorname{sp}(\phi(a)) = \operatorname{sp}(a)$ for any $a = a^*$. Further, $||\phi(a)|| = r(\phi(a)) = r(a) = ||a||$ for any such element. For an arbitrary elment b,

$$||\phi(b)||^2 = ||phi(b)^*\phi(b)|| = ||\phi(b^*b)|| = ||b^*b|| = ||b||^2$$

We conclude that ϕ is an isometry. Therefore, its image is closed since it is an isometry.
In the general case, assume that ϕ is not injective.
Consider the quotient algebra with $I = \ker\phi$ and $\psi([a]) = \phi(a)$.

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \phi(a) \subseteq B \\ & A/I \arrow[ur, "\psi"'] & \end{tikzcd}$$

We hae that π is a surjective $*$-homomorphism, ψ is an injective $*$-homomorphism, and A/I is a C^*-algebra. Therefore $A/I \cong \phi(A)$ is an isometric isomorphism as above.

Corollary

If $B \subseteq A$ are C^*-algebras with I a closed $*$-ideal of A, then $B + I$ is a $*$-subalgebra of A.

$$(B + I)/I \cong B/(B \cap I)$$

Proof

Consider the quotient map $\pi : A \to A/I$ restricted to B, $\pi|_B : B \to (B + I)/I$ which is closed in A/I. Therefore $B + I$ is closed in A because if $b_n + i_n \to a$, $[b_n] \to [a]$. Therefore $[a] \in (B + I)/I$ and $a = b + i \in B + I$.
We have that $B + I$ is a $*$-subalgebra of A where $b \mapsto [b]$ is surjective ($[b + i] = [b]$) $*$-homomorphism.
Then $\ker\pi|_B = B \cap I$, and $\hat{\pi} : B/(B \cap I) \to (B + I)/I$ is an isometric $*$-isomorphism.

Section 3.6: Positive Linear Functionals

Definition: Positive Linear Functional

A linear functional ϕ (on a C^*-algebra) is called positive ($\phi \geq 0$) if for all positive ($a \geq 0$) $a \in A$, $\phi(a) \geq 0$.
If, in addition, $\phi(e) = 1$, then ϕ is called a state.

Remark

Positive functionals satisfy $\phi(a^*) = \overline{\phi(a)}$.

Proof

For $a = a^*$, $a = a_+ - a_-$, $a_\pm \geq 0$. Therefore $\phi(a) = \phi(a_+) - \phi(a_-) \in \mathbb{R}$.
For general $a = s + it$, s and t self-adjoint,

$$\phi(a^*) = \phi(s - it) = \phi(s) - i\phi(t) = \overline{\phi(s) + i\phi(t)} = \overline{\phi(a)}$$

Remarks

- $\phi_1, \phi_2 \geq 0$ implies $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$ if and only if $\phi_1 - \phi_2 \geq 0$

Examples

$B = C(X)$ with X compact Hausdorff. $M(X)$ regular borel measures.

- All bounded linear functionals, μ a complex regular borel measure.

$$\phi_\mu(f) = \int_X f \, d\mu$$

- Positive linear functionals, μ a (positive) regular Borel measure

$$\phi_\mu(f) = \int_X f \, d\mu$$

- State $\phi(e) = 1$ if and only if $\mu(x) = \int_X \, d\mu = 1$.
- Multiplicative linear functionals, δ_{x_0} the Dirac point measure.

$$\phi_\mu(f) = f(x_0) = \int f \, d\delta_{x_0}$$

- $[f, g] = \int_X \overline{f} g \, d\mu$ semi-inner product (not a norm since $[f, f] = 0$ does not imply $f = 0$, and no completeness).

November 27, 2024

Recall: Positive Linear Functionals

C^*-algebra A, $\phi \geq 0$ if $\phi(a) \geq 0$, $\forall\, a \in A$, $a \geq 0$.

- $\phi \geq 0$ gives $\overline{\phi(a)} = \phi(a^*)$
- $\phi_1, \phi_2 \geq 0$ implies $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$ if and only if $\phi_1 - \phi_2 \geq 0$

Example

$A = C(X)$ positive linear functions, $\phi(f) = \int_X f\, d\mu$, μ regular Borel measure.
ϕ is a state if $\phi(e) = 1$

Example

$A = L(\mathbb{C}^n) = \mathbb{C}^{n\times n}$ linear functionals

$$\phi_B(A) = \mathrm{trace}(B^T A) = \sum_{i,j=1}^{n} b_{ij} a_{ij}$$

Positive linear functionals correspond to $B \geq 0$. For $A \geq 0$, this implies

$$\phi_B(A) = \mathrm{trace}(A^{1/2} B^T A^{1/2}) = \mathrm{trace}(B^T A)$$

where $A = A^{1/2} A^{1/2}$. $B \geq 0$ implies that

$$\phi_B(A) = \mathrm{trace}(A^{1/2}(B^T)^{1/2}(B^T)^{1/2}A^{1/2}) = \mathrm{trace}(C^* C) = \sum_{j,k} \overline{C_{jk}} C_{jk} \geq 0$$

Conversely, assume $\phi_B \geq 0$ and choose

$$A = xx^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \overline{x_1} & 7\cdots & \overline{x_n} \end{pmatrix}$$

Then

$$\phi_B(A) = \mathrm{trace}(B^T xx*) = x^* B^T x = \left\langle B^T x, x \right\rangle$$

for any $x \in \mathbb{C}^n$. Therefore $B^T \geq 0$ and $B \geq 0$.

Example

$A = L(H)$. A subset of positive funcionals is described by positive trace class operators $\mathcal{C}_1(H)$.
For $K \in \mathcal{C}_1(H)$, $\phi_K(A) = \mathrm{trace}(KA)$.

Finite Rank Operators

Finite rank operators are a subset of trace class operators. For $x \in H$, $K = x \otimes x^*$.
$kz = x \cdot \langle z, x \rangle$.

Proposition: Schwarz Inequality

For a positive linear functional $\phi \geq 0$, $|\phi(x^* y)| \leq \phi(x^* x) \cdot \phi(y^* y)$, $\forall x, y \in A$.

Proof

$x^* x \geq 0$ and $y^* y \geq 0$, so $\phi(x^* x) \geq 0$ and $\phi(y^* y) \geq 0$.
Define $[x, y] = \phi(y^* x)$ which has

- linearity in x and anti-linearity in y.
- $[y, x] = \overline{[x, y]}$ since $\phi(x^* y) = \phi((y^* x)^*) = \overline{\phi(y^* x)}$.
- $[x, x] = \phi(x^* x) \geq 0$.

Then $[x - \lambda y, x - \lambda y] \geq 0$ for all λ. Assume, without loss of generality, that $[x, y]$ is real (otherwise we may replace ($[\tau x, y]$ for $|\tau| = 1$) If $[x, x] = 0$ or $[y, y] = 0$, then $[x, y] = 0$. Otherwise, put $\lambda = -\frac{[x,y]}{[y,y]}$.
If $[x, x] = [y, y] = 0$, then $-\lambda[x, y] + \overline{\lambda}[y, x] = 0$ implies that $[x, y] = 0$.

Rmeark

$p(x) = \sqrt{[x, x]}$ is only a seminorm, not a norm. That is $p(x) = 0 \;/\; x = 0$.

Proposition:

If $\phi \geq 0$, then ϕ is bounded and $||\phi|| = \phi(e)$.

Proof

$$|\phi(e^* x)|^2 \leq \phi(x^* x) \cdot \phi(e^* e)$$

We know that $r(x^* x) = ||x^* x|| = ||x||^2$ and $\mathrm{sp}(x^* x) \subseteq [0, ||x||^2]$.
So $||x||^2 \cdot e - x^* x \geq 0$ ($x^* x \leq ||x||^2 \cdot e$) and $\phi(||x||^2 \cdot e - x^* x) \geq 0$.

$$\phi(e) \cdot ||x||^2 \geq \phi(x^* x)$$

Therefore $|\phi(x)|^2 \leq ||x||^2 \cdot \phi(e)^2$ implies that $\phi(e) \leq ||\phi|| \leq \phi(e)$ which shows also that ϕ is bounded.

Remark

$\phi \geq 0$ and $\phi(e) = 0$ implies that $\phi \equiv 0$.

Proposition

If ϕ is a bounded linear functional on A and $\phi(e) = ||\phi||$ then $\phi \geq 0$.

Proof

Without loss of generality, consider $\phi(e) = 1$.
Assume that $a \geq 0$ and $\phi(a) = \alpha + i\beta < 0$.
If $\beta \neq 0$, them consider $\phi(a + i\lambda e) = \alpha + i(\beta + \lambda)$ for $\lambda \in \mathbb{R}$.

$$
\begin{aligned}
|\beta+\lambda| &\leq |\phi(a+i\lambda e)| \\
&\leq ||a+i\lambda e|| \\
&= ||(a+i\lambda e)^*(a+i\lambda e)||^{1/2} \\
&= ||a^*a+a^*i\lambda - ai\lambda + \lambda^2 e||^{1/2} \\
&\leq (||a||^2+\lambda^2)^{1/2}
\end{aligned}
$$

So $(\beta+\lambda)^2 \leq ||a||^2+\lambda^2$ which means $\beta^2+2\beta\lambda \leq ||a||^2$ which cannot hold for $\lambda \to \pm\infty$.
If $\beta = 0$ and $\alpha < 0$, then

$$\phi(||a||e-a) = ||a|| - \alpha > ||a||$$

On the other hand $\phi(||a||e-a) \leq ||||a||e-a|| \leq ||a||$ since $\mathrm{sp}(a) \subseteq [0, ||a||]$ implies that $\mathrm{sp}(||a||e-a) \subseteq [0, ||a||]$ and $||||a||e-a|| = r(||a||e-a|| \leq ||a||$ which is also a contradiction.

Theorem:

Let B be a C^*-subalgebra of a C^*-algebra A (all unital) and ϕ be a positive linear functional on B.
Then ϕ has an extension to a positive linear functional on A.

Proof

$\phi \geq 0$ (on B), so $\phi \in B^*$ and $||\phi|| = \phi(e)$.
Hahn-Banach tells us that there exists an extension $\psi \in A^*$ onto A with $\psi|_B = \phi$ such that $||\psi|| = ||\phi||$. So $||\psi|| = \phi(e) = \psi(e)$. By the preceding proposition, $\psi \geq 0$ on A.

Section 3.7: Representations of C*-Algebras and GNS-Construction

Definition: Representation of a C*-Algebra

A representation (π, H) of a C^*-algebra A is a $*$-homomorphism $\pi : A \to L(H)$ such that $\pi(e) = I$.

Example

$A = L(H)$, $\pi = \mathrm{id}$ is trivial $[\mathrm{id}, H]$.

Definition: Inflation

If (π, H) is a representation of A, the inflation π^n is given by the block matrix

$$\pi^n(a) = \begin{pmatrix} \pi(A) & & & \\ & \pi(A) & & \\ & & \ddots & \\ & & & \pi(A) \end{pmatrix}$$

with $H_n = H \oplus H \oplus \cdots \oplus H$ $(= H \times H \times \cdots \times H)$.

Example

$A = C(X)$ (arbitary μ Borel measure) or $A = L^\infty(X, d\mu)$.
Take $H = L^2(X, d\mu)$, then $\pi : C(X) \ni f \mapsto M_f \in L(H)$ by $M_f g = fg$ where $g \in L^2(X, d\mu)$.

Definition: Direct Sum of Representations

The direct sum of representations (π_n, H_n) by $\pi = \bigoplus_n \pi_n$ and $H = \bigoplus_n H_n$.
For a finite or countable direct sum, we may think of this as

$$\pi(a) = \begin{pmatrix} \pi_1(a) & & & \\ & \pi_2(a) & & \\ & & \ddots & \\ & & & \pi_n(a) \end{pmatrix}$$

Definition: Equivalence of Representations

Two representations (π_1, H_1) and (π_2, H_2) are said to be equivalent if there exists unitary $U : H_1 \to H_2$ ($U^* U = I_{H_1}$ and $UU^* = I_{H_2}$) such that $U\pi_1(a)U^* = \pi_2(a)$.

Definition: Cyclic Representation

A representation (π, H) is cyclic if there exists some $v \in H$ such that $\mathrm{clos}(\pi(A)v) = H$ (i.e. $\mathrm{clos}\{\pi(a)v \,:\, a \in A\} = H$).

Example

Take

$$A = \left\{ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right\}$$

with $\pi(a) = a \in L(\mathbb{C}^2)$ sinc by $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ for $v_1, v_2 \neq 0$.
If instead we take

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$

and $\pi(a) = a$ then this is not cyclic since $\pi(A)v = \mathrm{lin}\{v\}$.

Theorem

Each representation is equivalent to a direct sum $\bigoplus_n \pi_n$ of cyclic representations.

Proof

If π is not cyclic, show that $\pi \cong \pi_1 \oplus \pi_2$.
Apply Zorn's lemma.

December 2, 2024

Recall

- Representation (π, H)
  - $\pi : A \to L(H)$ a $*$-homomorphism.

- Direct Sums
  - $\pi = \bigoplus \pi_\omega$ and $H = \bigoplus H_\omega$
- Cyclic (π, H)
  - $\exists v \in H$ such that $\mathrm{clos}(\pi(A)v) = H$.
- Theorem: Every representation is equivalent to a direct sum of cyclic representations.
  - $(\pi_1, H_1) \sim (\pi_2, H_2)$ if $\pi_1(a) = U\pi_2(a)U^*$ for $U : H_2 \to H_1$.

Proof

Let $\mathcal{E} = \{E \subseteq H : \pi(A)v_1 \perp \pi(A)v_2,\ \forall v_1, v_2 \in E,\ v_1 \neq v_2\}$ which may be ordered by inclusion to apply Zorn's lemma. Then $\mathcal{E}$ admits a maximal element $E \in \mathcal{E}$. Define $H_0 = \bigoplus_{v\in E} \mathrm{clos}\{\pi(A)v\} \subseteq H$. Note that H_0 is closed, since it is an orthogonal sum.
Claim: $H_0 = H$, otherwise $H = H_0 \oplus H_1$ for $H_1 \neq \{0\}$. Then for every $v \in H_0$, $h \in H_1$ and $a, b \in A$, $\langle \pi(A)v, h\rangle = \left\langle \pi(b^* a)v, h\right\rangle = 0$. Therefore

$$\begin{aligned} 0 &= \langle \pi(a)v, \pi(b)h\rangle \\ &\pi(a)v \perp \pi(b)h \\ &\pi(A)v \perp \pi(A)h \end{aligned}$$

This means that $E \cup \{h\} \in \mathcal{E}$, but E was maximal by assumption so $h \in E$. This implies that $H_1 = \{0\}$, a contradiction.
Now $H_v = \mathrm{clos}(\pi(a)v)$ for $v \in E$, so $H_{v_1} \perp H_{v_2}$ for $v_1 \neq v_2$, $H = \bigoplus_{v\in E} H_v$, and $\pi_v(a) = \pi(a)|_{H_v}$ is a representation.
$\pi(a)H_v \subseteq H_v$ and $\pi(a^*)H_v \subseteq H_v$.
For $H = H_{v_1} \oplus H_{v_2}$, with $\left\{\binom{*}{0}\right\} = H_{v_1}$,

$$\pi(a) = \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Then, since $\pi(a^*) = \pi(a)^* = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$,

$$\pi(a) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

with respect to $H_{v_1} \oplus H_{v_2}$. Therefore $\pi(a) \cong \bigoplus_{v\in E} \pi_v(a)$ and $H = (\dot{+})\, H_v \cong \bigoplus H_v$ (where the former is a literal sum). Write

$$v = \sum_{b_v \in H_v} b_v \to (b_v)_{v\in E}$$

and $||v|| = ||\sum b_v|| = \left(\sum ||b_v||^2\right)^{1/2} = ||(b_v)||_{\bigoplus H_v}$.

GNS-construction

Gelfand-Naimark-Segal

Theorem

For a unital C^*-algebra A:

1. for a positive linear functional ϕ, there exists a cyclic representation (π_ϕ, H_ϕ) and a cyclic vector v_ϕ such that

$$\phi(a) = \langle \pi_\phi(a) v_\phi, v_\phi \rangle$$

2. For a cyclic representation (π, H) with cyclic vector $v \in H$ and positive linear functional $\phi(a) := \langle \pi(a)v, v\rangle$, we have that $(\pi, H) \sim (\pi_\phi, H_\phi)$.

Remark

Positive linear functionals are in one to one correspondence with equivalence classes of cyclic representations.

Proof of A

Write $[a, b] = \phi(b^* a)$, a semi-inner product ($[a, a] = 0 \not\Longrightarrow a = 0$).
Take $L = \{a \in A : \phi(a^* a) = 0\}$ a closed linear subspace. Since $|\phi(b^* a)|^2 \le \phi(a^a a) \cdot \phi(b^* b)$, for each $a, b \in L$ we have that $[a, b] = 0$ implies that $[a + b, a + b] = 0$.
L is a left-ideal. That is, $a \in A$ and $x \in L$ implies $ax \in L$. Write

$$\phi((ax)^* ax)^2 = \phi(x^* a^* ax)^2 \le \phi((x^* a^* a)^* x^* a^* a) \cdot \underbrace{\phi(x^* x)}_{=0}$$

For $H_0 = A/L$ a vector space with well-defined inner product $\langle a + L, b + L\rangle = [a, b] = \phi(b^* a)$, we have an induced norm $||a + L||^2 = \phi(a^* a)$ and can take H_ϕ to be the completion of H_0. Then

$$||ax + L||^2 = \phi(x^* a^* ax) \le ||a||^2 \cdot ||x + L||^2$$

Write $\pi_\phi(a) : H_0 \to H_0$ by $x + L \mapsto ax + L$ (which is well-defined because L is a left-ideal). This map is linear, bounded (by previous computation), and may be extended by continuity to $\pi_\phi(a) : H_\phi \to H_\phi$.
It remains to show that $\pi_\phi, H_\phi)$ is a representation. It is multiplicative $\pi_\phi(ab) = \pi_\phi(a)\pi_\phi(b)$ from the definition. We have also that $\pi_\phi(a^*) = \pi_\phi(a)^*$ since

$$\begin{aligned}
\langle \pi_\phi(a)^*(x + L), y + L\rangle_{H_0} &= \langle x + L, \pi_\phi(a)(y + L)\rangle \\
&= \langle x + L, ay + L\rangle \\
&= \phi((ay)^* x) \\
&= \phi(y^*(a^* x)) \\
&= \langle a^* x + L, y + L\rangle \\
&= \langle \pi_\phi(a^*)(x + L), y + L\rangle_{H_0}
\end{aligned}$$

Finally, we have cyclic vector $v_\phi = e + L$ since $\pi_\phi(a)(e + L) = ae + L = a + L$. So $\pi_\phi(A)(e + L) = A/L = H_0$ and the closure is H_ϕ.

Definition: State space of a C*-Algebra

The state space of an algebra A is $\mathcal{S} = \{\phi \ge 0 : \phi(e) = 1\}$ for positive linear functionals ϕ.

Remark

$||\phi|| = \phi(e) = 1$, so $\mathcal{S}_A$ is a subset of the unit ball of the dual space A' of A. This is convex and compact in the weak∗-topology.
If A is commutative, then $\mathcal{S}_A$ is a maximal ideal space.

Definition: Extreme Points of a Convex Set

The point $a \in C$, for C convex, is extreme if $a = \lambda a_1 + (1-\lambda)a_2$, $a_1, a_2 \in C$, $\lambda \in (0,1)$ implies that $a_1 = a_2 = a$.
Write $\mathrm{ex}(C)$ for the set of extreme points.

IMAGE 1

The collection $\mathrm{ex}(\mathcal{S}_A)$ are called the pure states.

Theorem: Krein-Milman

For a compact, convex set C of a (Hausdorff, locally compact) topological vector space such as $C = \mathcal{S}_A$, $\mathrm{ex}(S_A) \neq 0$ and $\mathcal{S}_A = \mathrm{clos}(\mathrm{conv}(\mathrm{ex}(\mathcal{S}_A)))$ (i.e. the closure of the convex hull of the extreme points).

Proposition:

$||a|| = \max\{\phi(a) : \phi \in \mathcal{S}_A\}$ for all $a \geq 0$.

Proof

Case $A = C(X)$ for X a compact, Hausdorff space, $\mathcal{S}_A$ can be identified with the positive regular Borel measures μ on X with $\mu(X) = 1$.

$$\phi_\mu(a) = \int_X a(x)\, d\mu(x)$$

The extreme points are given by the Dirac measures $\mathrm{ex}(\mathcal{S}_A) \cong \{\delta_{x_0} : x_0 \in X\}$. Then for all $\phi \in \mathcal{S}_A$

$$|\phi(a)| \leq \underbrace{||\phi||}_{=\phi(e)=1} = ||a||$$

So

$$||a||_A = \max_{x \in X}\{a(x)\} = a(x_0) = \phi_{\delta_{x_0}}(a)$$

for some $x_0 \in X$.
In the general case, consider $a \geq 0$ and $A_0 = \mathrm{alg}\{e, a\}$ a commutative C^*-algebra. By what we have just proved, for some $\phi_0 \in \mathcal{S}_{A_0}$,

$$||a|| = \phi_0(a)$$

We may extend $\phi_0 \in \mathcal{S}_{A_0}$ to $\phi \in \mathcal{S}_A$ by Hahn-Banach for positive linear functionals such that

$$||a|| = \phi_0(a) = \phi(a)$$

Theorem

Let A be a unital C^*-algebra. Then there exists a representation (π, H) such that $\pi : A \to L(H)$ is a $*$-isometry.
If A is separable, then H can be chosen to be separable.

Proof

$F \subseteq \mathcal{S}_A$ weak*-dense (A separable implies the existence of a countable F).
Using the GNS-construction, $\phi \in F$ gives (π_ϕ, H_ϕ). Take $\pi = \bigoplus_{\phi\in F} \pi_\phi$ and $H = \bigoplus_{\phi\in F} H_\phi$.
$\pi : A \to L(H)$ is a $*$-homomorphism, since it is the direct sum of $*$-homomorphisms.

- $||\pi(a)|| \leq ||a||$
- $||\pi_\phi(a)||^2 \geq ||\pi_\phi(a)v_\phi||^2 = \langle \pi_\phi(a^*a)v_\phi, v_\phi\rangle = [a^*a + L, e + L] = \phi(a^*a)$.

$$||\pi(a)||^2 = \sup_{\phi\in F} ||\pi_\phi(a)||^2 \geq \sup_{\phi\in F} \phi(a^*a) = \max_{\phi\in\mathcal{S}_A} \phi(a^*a) = ||a^*a|| = ||a||^2$$

so $||\pi(a)|| = ||a||$ and π is an isometry.

Corollary

Every unital C^*-algebra is isometrically $*$-isomorphic to some closed $*$-subalgebra of some $L(H)$.

December 4, 2024

Chapte 4: Toeplitz Operators

$L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$; $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$.
$\phi : f(t) = \sum_{n\in\mathbb{Z}} f_n t^n \mapsto \{f_n\}_{n\in\mathbb{Z}}$ is an isometric isomorphism.
$f_n = \frac{1}{2\pi} f(e^{ix})e^{-inx}\, dx$ the fourier coefficients.

$$\{f \in L^2(\mathbb{T}) : f_n = 0,\ \forall n < 0\} = \left\{\sum_{n\geq 0}^{\infty} f_n t^n : \{f_n\} \in \ell^2(\mathbb{N}),\ |t| = 1\right\} = H^2(\mathbb{T}) \cong \ell^2(\mathbb{N})$$

With $H^2(\mathbb{T})$ the Hardy space. One can identify $f \in H^2(\mathbb{T})$ wit h $\hat{f}(z) = \sum_{n\geq 0} f_n z^n$ for $|z| < 1$ and an analytic extension $H^2(\mathbb{T}) \ni f \mapsto \hat{f}$ which is the set of all functions $\hat{f} : \mathbb{D} \to \mathbb{C}$ analytic and

$$\sup_{0\leq r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{f}(re^{ix})\right|^2 dx\right)^{1/2} < +\infty$$

More General

Take $L^p(\mathbb{T}) : \int_{-\pi}^{\pi} |f(e^{ix})|^p\, dx < +\infty$, for $1 < p < +\infty$.
Then $H^p(\mathbb{T}) = \{f \in L^p : f_n =,\ \forall n < 0\}$ is a closed subspace and it can be identified with the space of all $\hat{f} : \mathbb{D} \to \mathbb{C}$ analytic and

$$\sup_{0\leq r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{f}(re^{ix})\right|^p dx\right)^{1/p} < +\infty$$

Harmonic Extension / Analytic Extension

Write the harmonic extension

$$h_r : f = \sum_{n\in\mathbb{Z}} f_n t^n \in L^p(\mathbb{T}) \mapsto \sum_{n\in\mathbb{Z}} f_n r^{|n|} e^{inx}$$

where

$$t \mapsto \begin{cases} re^{ix} & n \geq 0 \\ \frac{1}{r}e^{ix} & n < 0 \end{cases}$$

Then $(h(f))(re^{ix}) = (h_r f)(e^{ix})$ and

$$\begin{aligned} t^n &\mapsto z^n, \quad n \geq 0 \\ t^n &\mapsto \overline{z}^{-n}, \quad n < 0 \end{aligned}$$

So $h(f)$ is harmonic as the sum of analytic and antianalytic functions, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h(f) = 0$. So

$$(h_r f)(e^{ix}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - r\cos(x-y) + r^2} f(e^{iy})\, dy$$

$P : L^p \ni \sum_{n\in\mathbb{Z}} f_n t^n \mapsto \sum_{n\geq 0} f_n r^n e^{inx} = \sum_{n\geq 0} f_n z^n$.

$$(Pf)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{t - z}\, dt \overset{t=e^{iy}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{iy})}{1 - e^{-iy} z}\, dy$$

$P_- : \sum_{n\in\mathbb{Z}} f_n t^n \mapsto \sum_{n\geq 0} f_n r^n e^{inx}$.

$$(P_- f)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{iy})}{1 - e^{+iy}\overline{z}}\, dy$$

So for $z = re^{ix}$,

$$\left(\frac{1}{1 - e^{-iy} z} + \frac{1}{1 - e^{+iy}\overline{z}}\right) = \frac{1 - r^2}{1 - r\cos(x-y) + r^2}$$

One can show for $1 \leq p < +\infty$, $f \in L^p$ implies that $h_r f \to f$ in the L^p-norm.
Then also if $f \in H^p$ then $h_r f \to f$.

Riesz Projection Operator

$P : \sum_{n\in\mathbb{Z}} f_n t^n \mapsto \sum_{n\geq 0} f_n t^n$ is bounded on L^p (for $1 < p < +\infty$, but not for $p = 1$ or $p = \infty$).
$H = P - P_-$ is the Hilbert transform on $\mathbb{T}$. So

$$h_r(Hf) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{-iy} re^{ix}} - \frac{1}{1 - e^{+iy} re^{-ix}} \right) f(e^{iy})\, dy$$

So as $r \to 1$,

$$(Hf)(e^{ix}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\Big(\frac{x-y}{2}\Big) f(e^{iy})\, dy$$

which is a singular integral and requires a Cauchy principle value.
It follows that $\operatorname{im} P = H^p(\mathbb{T})$ and $\ker P = \{f \in L^p \,:\, f_n = 0,\ \forall n \geq 0\} = \frac{1}{t}\overline{H^p(\mathbb{T})}$ where $\overline{H^P(\mathbb{T})} = \{f \in L^P(\mathbb{T}) \,:\, f_n = 0,\ \forall n > 0\}$.

Multiplication Operators

Take $a \in L^\infty(\mathbb{T})$, then

$$M(a) : L^p \ni f \mapsto af \in L^p$$

And

$$||M(a)f||_{L^p} = ||af||_{L^p} \leq ||a||_{L^\infty} \cdot ||f||_{L^p}$$

so $||M(a)|| \leq ||a||_{L^\infty}$.

Theorem

Let $A \in \mathcal{L}(L^p)$ be continuous and assume that $M(t)A = AM(t)$. Then there exists a unique $a \in L^\infty(\mathbb{T})$ such that $A = M(a)$ and $||M(a)|| = ||a||_{L^\infty}$.

Corollary

For $a \in L^\infty$, $M(a)$ is invertible if and only if a is invertible in L^∞.

Proof

$A = M(a)$ implies that $M(t)A = AM(t)$, so $A^{-1}M(T) = M(T)A^{-1}$. So $A^{-1} = M(b)$ for $b \in L^\infty$. Therefore

$$I = M(a)M(b) = M(ab)$$

implies $ab = I$.

Toeplitz Operators on Hardy Spaces

For $a \in L^\infty(\mathbb{T})$,

$$T(a) : H^p \ni f \mapsto P(\underbrace{af}_{\in L^p}) \in H^p$$

Note: $||T(a)|| \leq ||P|| \cdot ||M(a)|| = ||P|| \cdot ||a||_{L^\infty}$ so for $p = 2$, $||P|| = 1$ and for $p \neq 2$, $||P|| > 1$. Also $||a|| \leq ||T(a)||$.
$V_1 = T(t)$ by $f(t) \mapsto tf(t)$ and $V_{-1} = T(t^{-1})$ by $f(t) \mapsto \frac{f(t)-f(0)}{t}$ are the forward and backward shift respectively.
Then $T(a) = v_{-1}T(a)V_1$.

Theorem

Let $A \in \mathcal{L}(H^p)$ be bounded and assume that $V_{-1}AV_1 = A$. Then there exists a unique $a \in L^\infty(\mathbb{T})$ such that $A = T(a)$ and $||a|| \le ||T(a)|| \le ||P|| \cdot ||a||$.

Theorem

If $T(a)$ is invertible on H^p for $a \in L^\infty$, then a is invertible in L^∞. The converse is not true.

Laurent Operators

On $\ell^p(\mathbb{Z})$. If $p = 2$, then $L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$.
Take $a \in L^\infty$ by $a = \sum a_n t^n$. Then

$$M(a)f = af = \left(\sum_n a_n t^n\right)\left(\sum_m f_m t^m\right) = \sum_{n,m} a_n f_m t^{n+m} = \sum_k \left(\sum_n a_n f_{n-k}\right) t^k$$

The Laurent operators on $\ell^p(\mathbb{Z})$ are given by

$$L(a)x = y \quad y_n = \sum_{k \in \mathbb{Z}} a_{n-k} x_k \quad y = a * x$$

where $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ are in $\ell^p(\mathbb{Z})$.

Multiplier Classes

For $1 \le p \le \infty$, M_p is the set of all $\hat{a} \in L^1(\mathbb{T})$ with $a = (a_n)_{n\in\mathbb{Z}}$
For all finite supported $x = (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, $a * x \in \ell^p$ and $||a * x||_{\ell^p} \le C||x||_{\ell^p}$.
Then for $a \in M_p$ we can obtain $L(a)$ by continuous extension $x \in \ell^p$, $x^{(n)} \underset{\ell^p}{\to} x$, $L(a)x = \lim_{n\to\infty} a * x^{(n)}$ in the ℓ^p-norm.
Now $L(a)x = a * x$ the convolution product converges. So for $\ell^p(\mathbb{Z})$

$$x = \{x_n\}_{n\in\mathbb{Z}} \cong \begin{pmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

and

$$L(a) = \begin{pmatrix} \ddots & & & & & \\ a_1 & a_0 & a_{-1} & & & \\ & a_1 & a_0 & a_{-1} & & \\ & & a_1 & a_0 & a_{-1} & \\ & & & & & \ddots \end{pmatrix}$$

which is $(a_{j-k})_{j,k\in\mathbb{Z}}$. For Laurent operators, $U_1 L(a) = L(a) U_1$ for

$$U_1 = L(t) \cong \begin{pmatrix} \ddots & & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Multiplier Classes

- $M_2 = L^\infty(\mathbb{T})$
- $M_p = M_q$ for $\frac{1}{p} + \frac{1}{q} = 1$
- $M_1 = M_\infty = W$ the Wiener class $\{a_n\} \in \ell^1$ with $M_1 \subseteq$

$P : \ell^p \ni \{x_n\}_{n\in\mathbb{Z}} \mapsto \{y_n\}_{n\in\mathbb{Z}} \in \ell^p$

$$y_n = \begin{cases} x_n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where $\operatorname{im} P \cong \ell^p(\mathbb{N})$ so $T(a)x = P(L(a)x)$ on $\ell^p(\mathbb{N})$ and

$$T(a) = \begin{pmatrix} a_0 & a_1 & & & \\ a_{-1} & a_0 & a_1 & & \\ & a_{-1} & a_0 & a_1 & \\ & & & & \ddots \end{pmatrix}$$

Remarks

- A is bounded on ℓ^p, $U_1 A = AU_1$ implies $A = L(a)$ with $a \in M_p$.
- $A = L(a)$, $a \in M_p$ invertible implies a is invertible in M_p.

Theorem

Let $A = \mathcal{L}(\ell^p(\mathbb{N}))$ be bounded and $V_{-1}Av_1 = A$. Then there exists a unique $a \in M_p$ such that $A = T(a)$ and $||T(a)|| = ||L(a)|| =: ||a||_{M_p}$.

Theorem

If $T(a)$ is invertible, then a is invertible in M_p. The converse is not true.

$$V_1 = \begin{pmatrix} 0 & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & \ddots \end{pmatrix} \quad V_{-1} = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

So $V_{-1}AV_1 = A$ gives a diagonal setup.

Theorem

For $a \in C(\mathbb{T})$, $T(a)$ is invertible on $H^p(\mathbb{T})$ if and only if $a(t) \neq 0$ for every $t \in \mathbb{T}$ and $\operatorname{wind}(a) = 0$.

Theorem (Widom)

The spectrum of $T(a)$ on $H^p(\mathbb{T})$ is connected.

Theorem

For $a \in L^{\infty}(\mathbb{T})$, a real valued on H^2,

$$\operatorname{sp} T(a) = [\alpha, \beta]$$

where $\alpha = \text{ess-inf}_{t \in \mathbb{T}}\, a(t)$ and $\beta = \text{ess-sup}_{t \in \mathbb{T}}\, a(t)$.

Example

For $a = \chi_E$, $\mu(E) > 0$, $\mu(\mathbb{T} \setminus E) > 0$, we have that $\operatorname{sp} T(\chi_E) = [0, 1]$.