Math 24 Discussion Section

Warm Up

Given a nonsingular matrix $A \in \mathbb{C}^{n\times n}$, clearly define in both symbols and plain language...

- the transpose of A.
- the conjugate of A.
- the adjoint (or conjugate transpose) of A.
- the inverse of A.
- the eigenvalues of A.
- the eigenvectors of A

Similarly, define what it means is A is...

- diagonal.
- upper/lower triangular.
- Hermitian (or self-adjoint)
- unitary.
- normal.

Problems

1. Transform $u'' + pu' + qu = g,\ u(0) = u_0,\ u'(0) = u_0'$ into an initial value problem for two first-order equations.
2. Consider the initial value problem

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1 + 2x_2, \quad x_1(0) = -2 \\ x_2' &= -2x_1 - \frac{1}{2}x_2, \quad x_2(0) = 2. \end{aligned}$$

   a. Transform the given system into a single equation of second order.
   b. Find x_1 and x_2 that also satisfy the initial conditions.
   c. Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

3. Verify that $x = (6e^{-t}, -8e^{-t} + 2e^{2t}, -4e^{-t} - 2e^{2t})^T$ satisfies

$$x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x.$$

4. Given

$$tx' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x\ (t > 0);\ x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1},\ x^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2,$$

   a. Show that $x^{(1)}$ and $x^{(2)}$ are solutions.
   b. Show that $x = c_1x^{(1)} + c_2x^{(2)}$ is also a solution for any values of c_1 and c_2.
   c. Show that $x^{(1)}$ and $x^{(2)}$ form a fundamental set of solutions.
   d. Compute $W[x^{(1)}, x^{(2)}](t)$.
   e. Show that $W := W[x^{(1)}, x^{(2)}](t)$ solves Abel's equation $W' = (p_{11}(t) + p_{22}(t))W$.

Matrix/Vector Problems

1. If $z \in \mathbb{C}$ and
$$A(z) = \begin{pmatrix} e^z & 2e^{-z} & e^{2z} \\ 2e^z & e^{-z} & -e^{2z} \\ -e^z & 3e^{-z} & 2e^{2z} \end{pmatrix},$$
compute A^T, $\overline{A}$, and A^*.

2. Solve or show that no solution exists for
$$\begin{aligned} x_1 - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

3. Find the eigenvalues and eigenvectors for
$$\begin{pmatrix} \frac{11}{9} & -\frac{2}{9} & \frac{8}{9} \\ -\frac{2}{9} & \frac{2}{9} & \frac{10}{9} \\ \frac{8}{9} & \frac{10}{9} & \frac{5}{9} \end{pmatrix}.$$

4. Let V be a vector space over some field F and let $a, b \in F$ while $x, y, z \in V$. We say that V is an inner product space if it is equipped with a function $\langle \cdot, \cdot \rangle : V \times V \to F$ satisfying the following properties.

   - Linearity in the First Term: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
   - Conjugate Symmetry: $\overline{\langle x, y \rangle} = \langle y, x \rangle$.
   - Positive Definiteness: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

   Show that for any fixed n, the n-dimensional vector space $\mathbb{C}^n$ over the field $\mathbb{C}$ of complex numbers is an inner product space when equipped with the function $(x, y) := \sum_{i=1}^{n} x_i \overline{y_i}$.

   Show also that linearity and conjugate symmetry lead to (x, y) being antilinear in the second term.
$$(x, ay + bz) = \overline{a}(x, y) + \overline{b}(x, z).$$

   Observe that restricting (x, y) to the real case gives bilinearity (i.e. linearity in both terms). The complex case is referred to as sesquilinearity.