Question of the Day

If you could pick two animals and swap the noises they make, which and why?

On the Docket

Check-in
Concept Review: Determinants
Concept Review: Subspaces
Concept Review: Abstract Vector Spaces

We know that we can calculate the determinant of a matrix by cofactor expansion, so let us look at a concrete example.
For an arbitrary 3×3 matrix, the determinant is a function which returns

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdg - ceg$$

Geometrically, this is the oriented $(+/-)$ rvolume of a parallelepiped which is described by three 3-vectors.
Consider what happens if we set one of the vectors to zero.

$$\begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{vmatrix} = aei - a0h - bd0 + b0g + 0dg - 0eg$$

The determinant collapses to zero. The parallelogram, described by two vectors, has no volume in 3-space. In that space, it is "flat" and cannot be inverted. This behavior continues in higher dimensions.
We can play a similar game to see the effects of row operations.

A vector space V over a field $\mathbb{F}$ is a nonempty set equipped with two binary functions

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{F} \times V \to V$$

such that for every $\vec{u}, \vec{v}, \vec{w} \in V$ and every $a, b \in \mathbb{F}$ the following hold.

Vector Space Axioms

1. Commutativity of addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. Associativity of addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. Existence of an additive identity: $0 + \vec{u} = \vec{u} = \vec{u} + 0$
4. Existence of additive inverses: $\vec{u} + (-\vec{u}) = 0$
5. Distribution over scalar sums: $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
6. Distribution over vector sums: $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
7. Associativity of scalar multiplication: $a(b\vec{u}) = (ab)\vec{u}$
8. Existence of scalar multiplicative identity: $1\vec{u} = \vec{u}$

Subspace

For a vector space V over a field $\mathbb{F}$, a subset $W \subseteq V$ is a subspace if for each $\vec{u}, \vec{v} \in W$ and for every $\lambda \in \mathbb{F}$

1. W is closed under addition: $\vec{u} + \vec{v} \in W$
2. W is closed under scalar multiplication: $\lambda \vec{u} \in W$

Consider $\mathbb{R}_2[x]$, the set of all polynomials with real coefficients of degree two or less.

$$\mathbb{R}_2[x] = \{aX^2 + bX + c \,:\, a,b,c \in \mathbb{R}\} \simeq \left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \,:\, a,b,c \in \mathbb{R}\right\}$$

Just like vectors in $\mathbb{R}^3$, they are uniquely described by a collection of three real numbers and some consistent ordering system.
Similarly, we may examine 2×2 matricies with real valued entries.

$$\mathrm{Mat}_{2\times 2}(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, a,b,c,d \in \mathbb{R}\right\} \simeq \left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \,:\, a,b,c,d \in \mathbb{R}\right\}$$