Question of the Day

What is a number? Is ∞ a number?

On the Docket

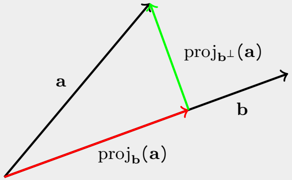
Check-in
Concept Review: Projection, Orthogonal Projection and the Dot Product
Concept Review: Cuachy-Schwarz and Triangle Inequalities
Concept Review: Linear Transformations
Concept Review: Null Space

Vector Projection

The projection of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$ is given equivalently by

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = (||\mathbf{a}||\cos(\theta))\hat{\mathbf{b}} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$$

where $\hat{\mathbf{b}}$ is a unit vector (magnitude 1) in the direction of $\mathbf{b}$.



Orthogonal Projection

The orthogonal projection of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}^\perp}(\mathbf{a}) = \mathbf{a} - \text{proj}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$$

Dot Product

The dot product of two n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta) = \sum_{i=1}^{n} a_i b_i$$

We can use the geometric description of the dot product to calculate the angle between two vectors

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta) \iff \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \cos(\theta) \iff \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}\right) = \theta.$$

Two vectors are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. This happens precisely when $\theta = \pm 90^\circ$ (or when either $\mathbf{a}$ or $\mathbf{b}$ are the zero vector).

Exercise

Find all vectors in $\mathbb{R}^4$ orthogonal to

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 1 \\ -2 & 0 & -6 \end{pmatrix}$$

The Cauchy-Schwarz Inequality

For vectors $\mathbf{a}$ and $\mathbf{b}$

$$||\mathbf{a} \cdot \mathbf{b}|| \leq ||\mathbf{a}||||\mathbf{b}||$$

with equality when $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.

The Triangle Inequality

For vectors $\mathbf{a}$ and $\mathbf{b}$

$$||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$$

with equality $\mathbf{a} = \lambda\mathbf{b}$ for some real $\lambda > 0$.

Linear Transformations

The function $T : X \to Y$ is said to be a linear function / transformation / map if

1. For all $x_1, x_2 \in X$, $T(x_1 + x_2) = T(x_1) + T(x_2)$ and
2. For any $x \in X$ and scalar c, $T(cx) = cT(x)$.

$T_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a linear map; so is $T_2(x) = x$.

Exercise

Show that T_1 and T_2 are linear transformations.

What about $T_3(x) = x^2$ or $T_4(x) = 3x + 1$?

Exercise

Compute the "standard matrix" (i.e. in terms of the standard basis) of

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_3 + x_4 \\ x_1 + x_4 \\ x_1 + x_2 \end{pmatrix}$$

Null Space

Let A be an $m \times n$ matrix. The null space of A, written $\mathrm{null}(A)$, is the collection of all n-vectors x in the space such that $Ax = 0$.

Theorem 3.2.3

The following are equivalent

1. The null space of A is trivial (i.e. contains only 0).
2. Every column of A is a pivot column.
3. The columns of A are linearly independent.

Exercise

Find the null space of

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & 3 & -1 & -3 \\ 3 & -4 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{pmatrix}$$