Question of the Day

If you had to replace your legs with the leg of another animal, which animal and why?

On the Docket

Concept Check-in: New Vocabulary
Concept Check-in: Math Vocabulary
Concept Review: Span
Concept Review: Linear Independence
Concept Review: Basis
Challenge Problems

In plain language, describe

- Set.
- Space (e.g. a vector space).
- Subset.
- Subspace.
- Linear span.
- Linear dependence.
- Linear independence.
- Column space.
- Row space.
- Trivial (e.g. a trivial solution).
- Basis.
- The standard basis (e.g. of $\mathbb{R}^3$).

Let $S = \{v_1, \ldots, v_k\}$ be a collection of vectors in $\mathbb{R}^n$.

Necessary

A condition which must occur to allow for some consequence.

Example: if $\mathrm{span}(S) = \mathbb{R}^n$, then it is necessary that $k \geq n$.

Sufficient

A condition which causses some consequence.

Example: $k > n$ is sufficient to show that S is linearly dependent.

Conditions may be necessary *and* sufficient.

Equivalent

Two or more statements are equivalent if they share the same truth value.

That is, one being true means they are all true; one being false means they are all false.

Let $S = \{v_1, \dots, v_k\}$ be a collection of vectors.

Span

The span of S, written $\mathrm{span}(S)$, is the collection of all linear combinations of the elements of S.

That is, all vectors that may be constructed as

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

where a_i are scalars.

Exercise

Form a group of n people.

In turn, each member will be responsible for constructing one vector such that the span of all vectors in the group is $\mathbb{R}^n$.

Can you do it with vectors with only one non-zero component? What about two? Three? n?

Let $S = \{v_1, \ldots, v_k\}$ be a collection of vectors.

Linear Independence

The set S is said to be linearly independent if the only scalars a_i which make

$$a_1 v_1 + \cdots + a_k v_k = 0$$

true are $a_i = 0$ for all $1 \leq i \leq k$. That is, the equation has only the trivial solution.

Exercise

Are the vectors that were picked in the previous exercise linearly independent?

Do we need to do Gaussian elimination to answer this question?

Let V be a nonzero subspace of $\mathbb{R}^n$ and $S = \{v_1, \ldots, v_k\}$ a collection of vectors in $\mathbb{R}^n$.

Basis

We say that S is a basis for V if

1. $\mathrm{span}(S) = V$ and
2. S is linearly independent.

Note that it is possible for $V = \mathbb{R}^n$ such that S is a basis for $\mathbb{R}^n$.

Exercise

Is the set of vectors that your group chose a basis for $\mathbb{R}^n$?

1. Let $S = \{v_1, \ldots, v_k\}$ be a collection of vectors in $\mathbb{R}^n$. If $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$, explain why $\operatorname{span}(S) \neq \mathbb{R}^n$.
2. a) Show that the columns are linearly independent.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 3 & 4 & 4 \end{pmatrix}$$

b) Assume that $\{v_1, v_2, v_3\}$ are linearly independent and set

$$w_1 = v_1 + 2v_2 + 3v_3,\ w_2 = 2v_1 + 3v_2 + 4v_3,\ w_3 = -v_1 + 2v_2 + 4v_2.$$

Show that $\{w_1, w_2, w_3\}$ is linearly independent.

c) Show that the columns are linearly dependent and find a non-trivial dependence relation.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

d) Set

$$u_1 = v_1 + 2v_2 + 3v_3,\ u_2 = 2v_1 + 3v_2 + 4v_3,\ u_3 = v_1 + v_2 + v_3,$$

and prove that $\{u_1, u_2, u_3\}$ is linearly dependent.