Math 145 Discussion Section

Warm Up

Concept Review

With your group, discuss the following terms from lecture. Try to come up with both a clear, plain language definition and a rigorous, mathematical definition with appropriate formulas and expressions.

Vector Valued Function	Matrix Exponential	Numerical Scheme
Uniform Continuity	Hölder Continuity	Lipschitz Continuity
(Forward / Backward) Euler Method	Heun Method	Crank-Nicolson Method
Trapezoidal Rule	Order of a Scheme	Error of a Scheme

Interlude: Big O, Little O, and Asymptotic Equivalence

Problems

1. Show that the initial value problem

$$\begin{cases} \dot{x} = x^2 + h(x) \\ x(0) = x_0 \end{cases},$$

   where $h(x) \geq 0$ for all x, has finite time blowup.

2. Consider the equation $\frac{dy}{dt} = Ay$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (a) Show that $e^{tA} = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

   (b) Write the transfer matrix B such that $By = y(t + \Delta t)$.

   (c) Write the approximate transfer matrix for the Forward Euler scheme such that $\hat{B}u_n = u_{n+1}$.

   (d) Write the approximate transfer matrix for the Heun scheme.

3. Consider the initial value problem $\dot{x} = x$, $x(0) = 1$.

   (a) Solve for $x(1)$ analytically.

   (b) Estimate $x(1)$ numerically using the Forward Euler method with step size $\Delta t = 10^{-n}$ for $n = 0, 1, 2, 3, 4$.

   (c) Plot the error $E(\Delta t) = |\hat{x}(1) - x(1)|$ as a function of Δt.

   (d) Plot $\log(E)$ against $\log(\Delta t)$. Does this corroborate the convergence rate $E(\Delta t) = \mathcal{O}(\Delta t)$?