

Math 145 Discussion Section

Warm Up

Concept Review

With your group, discuss the following terms from lecture. Try to come up with both a clear, plain language definition and a rigorous, mathematical definition with appropriate formulas and expressions.

Cardinality

Maximum / Minimum

(Outer) Measure

Power Set

Supremum / Infimum

Distance Function

Limit Superior / Limit Inferior

Interlude: Sets and Measures

Problems

1. For $A, B \in 2^{\mathbb{R}}$, define

$$\text{dist}(A, B) := \inf\{|b - a| : (a, b) \in A \times B\}.$$

- (a) Find disjoint, nonempty sets $A, B \in 2^{\mathbb{R}}$ such that $\text{dist}(A, B) = 0$.
Conclude that $(2^{\mathbb{R}}, \text{dist})$ is not a metric space.

- (b) Define $\nu^* : 2^{\mathbb{R}} \rightarrow [0, \infty]$, the Lebesgue outer measure on \mathbb{R} , by

$$\nu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} |b_n - a_n| : A \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}.$$

Show that if $\text{dist}(A, B) = 0$ then it is possible that $\nu^*(A \cup B) \neq \nu^*(A) + \nu^*(B)$.

- (c) Review Caratheodory's lemma (Theorem 3.2, lecture 14) in light of the above.

2. For $M_2(\mathbb{R})$, the space of all 2×2 real matrices, define

$$d(A, B) := \left(\sum_{i,j=1}^2 (a_{ij} - b_{ij})^2 \right)^{1/2}.$$

- (a) Show that $(M_2(\mathbb{R}), d)$ is a metric space (hint: recall Cauchy-Schwarz).

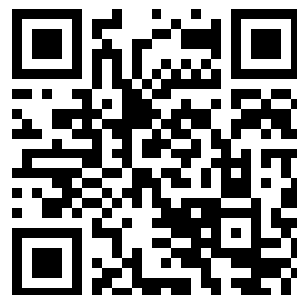
- (b) Show that the subset of invertible matrices, $GL_2(\mathbb{R})$, is not closed in $(M_2(\mathbb{R}), d)$.

3. Let $A \subset \mathbb{R}$ be bounded, and define $N(A, \varepsilon)$ to count the minimum number of closed intervals of length ε needed to cover A . Define the box-counting dimension of A , when it exists, to be

$$\dim_{\text{box}}(A) := - \lim_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{\log(\varepsilon)}.$$

- (a) Compute the box-counting dimension of the standard Cantor set (hint: let $\varepsilon_n = 3^{-n}$).

- (b) Show that $\dim_{\text{box}}(\mathbb{Q} \cap [0, 1]) = 1$.



Extra Problems

1. (Viana, Oliveira, ex. A.1.6. p. 442) (Lemma of Borel-Cantelli). Let $(E_n)_n$ be a countable family of measurable sets. Let F be the set of all points that belong to E_n for infinitely many values of n , that is,

$$F = \limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n.$$

Show that if $\sum_n \mu(E_n) < \infty$ then $\mu(F) = 0$.