Math 145 Discussion Section

Warm Up

Concept Review

With your group, discuss the following terms from lecture. Try to come up with both a clear, plain language definition and a rigorous, mathematical definition with appropriate formulas and expressions.

Cardinality	Power Set	
Maximum / Minimum	Supremum / Infimum	Limit Superior / Limit Inferior
(Outer) Measure	Distance Function	

Interlude: Sets and Measures

Problems

1. For $A, B \in 2^{\mathbb{R}}$, define

$$\text{dist}(A, B) := \inf\{|b - a| \ : \ (a, b) \in A \times B\}.$$

   (a) Find disjoint, nonempty sets $A, B \in 2^{\mathbb{R}}$ such that $\text{dist}(A, B) = 0$.
   Conclude that $(2^{\mathbb{R}}, \text{dist})$ is not a metric space.

   (b) Define $\nu^* : 2^{\mathbb{R}} \to [0, \infty]$, the Lebesgue outer measure on $\mathbb{R}$, by

$$\nu^*(A) := \inf\left\{\sum_{n=1}^{\infty} |b_n - a_n| \ : \ A \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)\right\}.$$

   Show that if $\text{dist}(A, B) = 0$ then it is possible that $\nu^*(A \cup B) \neq \nu^*(A) + \nu^*(B)$.

   (c) Review Caratheodory's lemma (Theorem 3.2, lecture 14) in light of the above.

2. For $M_2(\mathbb{R})$, the space of all 2×2 real matrices, define

$$d(A, B) := \left(\sum_{i,j=1}^{2} (a_{ij} - b_{ij})^2\right)^{1/2}.$$

   (a) Show that $(M_2(\mathbb{R}), d)$ is a metric space (hint: reacall Cauchy-Schwarz).

   (b) Show that the subset of invertible matrices, $GL_2(\mathbb{R})$, is not closed in $(M_2(\mathbb{R}), d)$.

3. Let $A \subset \mathbb{R}$ be bounded, and define $N(A, \varepsilon)$ to count the minimum number of closed intervals of length ε needed to cover A. Define the box-counting dimension of A, when it exists, to be

$$\dim_{\text{box}}(A) := -\lim_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{\log(\varepsilon)}.$$

   (a) Compute the box-counting dimension of the standard Cantor set (hint: let $\varepsilon_n = 3^{-n}$).

   (b) Show that $\dim_{\text{box}}(\mathbb{Q} \cap [0, 1]) = 1$.

Extra Problems

1. (Viana, Oliveira, ex. A.1.6. p. 442) (Lemma of Borel-Cantelli). Let $(E_n)_n$ be a countable family of measurable sets. Let F be the set of all points that belong to E_n for infinitely many values of n, that is,

$$F = \limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n.$$

Show that if $\sum_n \mu(E_n) < \infty$ then $\mu(F) = 0$.