

## Math 145 Discussion Section

### Warm Up

#### Concept Review

With your group, discuss the following terms from lecture. Try to come up with both a clear, plain language definition and a rigorous, mathematical definition with appropriate formulas and expressions.

Schwarzian Derivative  
Cantor Set

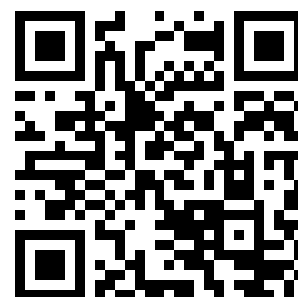
Schwarzian Min-Max Principle  
Immediate Basin of Attraction

Sharkovsky Ordering Theorem  
Operator vs Function

### Interlude: Differential Operators

#### Problems

1. Show that if  $Sf < 0$  and  $Sg < 0$  then  $S(g \circ f) < 0$ .
2. Consider the map  $f(x) = \pi \sin(x)$ .
  - (a) Identify the critical point  $p \in (0, \pi)$  and compute  $Sf(x)$  for any  $x \in [0, \pi] - \{p\}$ .
  - (b) Show that any immediate basin of attraction must have a boundary at  $0$ ,  $p$  or  $\pi$  (Hint: Schwarzian Min-Max Principle).
  - (c) Compute the forward orbit of  $p$ ,  $0$  and  $\pi$  under  $f$ , and evaluate  $f'$  at any eventual fixed points.
  - (d) Use (b) and (c) to conclude that  $f$  admits no attracting periodic orbits.
  - (e) Show that the restrictions  $f|_{[0,p]} : [0, p] \rightarrow [0, \pi]$  and  $f|_{[p,\pi]} : [p, \pi] \rightarrow [0, \pi]$  are monotonic and surjective.
  - (f) Conclude that all periodic points are repelling.
3. (Devaney, ex. 1 p. 95) Prove that the map  $f(x) = \pi \sin(x)$  is chaotic on the interval  $[0, \pi]$ .
  - (a) **Topological Transitivity** Let  $I \subset [0, \pi]$  be any open interval. Show that there must exist  $n \geq 1$  such that  $f^n(I) = [0, \pi]$ . Conclude that  $f^n(I) \cap U \neq \emptyset$  for any open set  $U \subset [0, \pi]$ .
  - (b) **Dense Periodic Points** Let  $I \subset [0, \pi]$  be any open interval. From (a), we have that  $I \subset f^n(I) = [0, \pi]$  for some  $n \geq 1$ . Use the Intermediate Value Theorem on  $g(x) = f^n(x) - x$  to show that there exists at least one point  $x \in I$  such that  $f^n(x) = x$ . Conclude that every open set  $U \subset [0, \pi]$  contains a periodic point.
  - (c) **Sensitivity to Initial Conditions** Fix  $\delta > 0$  and use (a) to show that for any point  $x \in [0, \pi]$  and neighborhood  $U \ni x$ , there exists  $y \in U$  and  $n \geq 1$  such that  $|f^n(x) - f^n(y)| > \delta$ .



## Extra Problems

Let  $\hat{\mathbb{C}}$  denote the Riemann sphere and  $C^n(\hat{\mathbb{C}})$  the space of all  $n$ -times continuously differentiable functions on  $\hat{\mathbb{C}}$ .

1. We say that  $f : \mathbb{C} \rightarrow \mathbb{C}$  is an affine transformation if  $f(z) = az + b$  for  $a, b \in \mathbb{C}$  and  $a \neq 0$ .
  - (a) For affine  $f$ , define a differential operator  $L : C^2(\hat{\mathbb{C}}) \rightarrow C^2(\hat{\mathbb{C}})$  such that  $Lf = 0$ .
  - (b) For affine  $g$ , compute  $\frac{d}{dz}(g \circ h)$  and  $\frac{d^2}{dz^2}(g \circ h)$ .
  - (c) Define a differential operator such that  $Pf = 0$  for affine  $f$  and  $P(g \circ h) = Ph$  for affine  $g$ .
2. We say that  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is a Möbius transformation if  $f(z) = \frac{az+b}{cz+d}$  for  $a, b, c, d \in \mathbb{C}$  and  $ad - bc \neq 0$ .
  - (a) Compute  $P(z^{-1})$ .
  - (b) Show that  $Sf := (Pf)' - \frac{1}{2}(Pf)^2 \equiv 0$  when  $f$  is a Möbius transformation.
  - (c) Show that  $S(g \circ f) = Sf$  when  $g$  is a Möbius transformation.