Math 145 Discussion Section

Warm Up

Concept Review

With your group, discuss the following terms from lecture. Try to come up with both a clear, plain language definition and a rigorous, mathematical definition with appropriate formulas and expressions.

Schwarzian Derivative	Schwarzian Min-Max Principle	Sharkovsky Ordering Theorem
Cantor Set	Immediate Basin of Attraction	Operator vs Function

Interlude: Differential Operators

Problems

1. Show that if $Sf < 0$ and $Sg < 0$ then $S(g \circ f) < 0$.
2. Consider the map $f(x) = \pi \sin(x)$.
   (a) Identify the critical point $p \in (0, \pi)$ and compute $Sf(x)$ for any $x \in [0, \pi] - \{p\}$.
   (b) Show that any immediate basin of attraction must have a boundary at 0, p or π (Hint: Schwarzian Min-Max Principle).
   (c) Compute the forward orbit of p, 0 and π under f, and evaluate f' at any eventual fixed points.
   (d) Use (b) and (c) to conclude that f admits no attracting periodic orbits.
   (e) Show that the restrictions $f|_{[0,p]} : [0, p] \to [0, \pi]$ and $f|_{[p,\pi]} : [p, \pi] \to [0, \pi]$ are monotonic and surjective.
   (f) Conclude that all periodic points are repelling.
3. (Devaney, ex. 1 p. 95) Prove that the map $f(x) = \pi \sin(x)$ is chaotic on the interval $[0, \pi]$.
   (a) **Topological Transitivity** Let $I \subset [0, \pi]$ be any open interval. Show that there must exist $n \geq 1$ such that $f^n(I) = [0, \pi]$. Conclude that $f^n(I) \cap U \neq \emptyset$ for any open set $U \subset [0, \pi]$.
   (b) **Dense Periodic Points** Let $I \subset [0, \pi]$ be any open interval. From (a), we have that $I \subset f^n(I) = [0, \pi]$ for some $n \geq 1$. Use the Intermediate Value Theorem on $g(x) = f^n(x) - x$ to show that there exists at least one point $x \in I$ such that $f^n(x) = x$. Conclude that every open set $U \subset [0, \pi]$ contains a periodic point.
   (c) **Sensitivity to Initial Conditions** Fix $\delta > 0$ and use (a) to show that for any point $x \in [0, \pi]$ and neighborhood $U \ni x$, there exists $y \in U$ and $n \geq 1$ such that $|f^n(x) - f^n(y)| > \delta$.

Extra Problems

Let $\hat{\mathbb{C}}$ denote the Riemann sphere and $C^n(\hat{\mathbb{C}})$ the space of all n-times continuously differentiable functions on $\hat{\mathbb{C}}$.

1. We say that $f : \mathbb{C} \to \mathbb{C}$ is an affine transformation if $f(z) = az + b$ for $a, b \in \mathbb{C}$ and $a \neq 0$.
   (a) For affine f, define a differential operator $L : C^2(\hat{\mathbb{C}}) \to C^2(\hat{\mathbb{C}})$ such that $Lf = 0$.
   (b) For affine g, compute $\frac{d}{dz}(g \circ h)$ and $\frac{d^2}{dz^2}(g \circ h)$.
   (c) Define a differential operator such that $Pf = 0$ for affine f and $P(g \circ h) = Ph$ for affine g.
2. We say that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation if $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.
   (a) Compute $P(z^{-1})$.
   (b) Show that $Sf := (Pf)' - \frac{1}{2}(Pf)^2 \equiv 0$ when f is a Möbius transformation.
   (c) Show that $S(g \circ f) = Sf$ when g is a Möbius transformation.