Teaching Assitant

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Office hours:

Mondays 11am-12pm - ARC 116
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Problem 1

Consider the rabbits vs sheep example $\dot{x} = x(3 - x - 2y)$, $\dot{y} = y(2 - x - y)$ for $x \geq 0$, $y \geq 0$. The goal is to use Lyapunov functions to trap the dynamics in a bounded region of space.

(a) Show that the function $L(x, y) = x^2 + y^2$ is strictly decreasing along the flow of the rabbit-sheep system, in the region $U^+ = \{x \geq 0, y \geq 0, x + 2y \geq 3, x + y \geq 2\}$. This means that if $x(t)$, $y(t)$ solves the rabbit-sheep system and $(x(t), y(t)) \in U^+$, then $\frac{d}{dt}(L(x(t), y(t))) < 0$.

(b) Draw the region $U^- = \{x \geq 0, y \geq 0, x + 2y \leq 3, x + y \leq 2\}$ and deduce from the previous question that every trajectory in U^+ eventually reaches the bounded region U^-.

(c) Can you cook up a similar argument with $L(x, y) = x + y$?

Problem 2

Fix $r = 1.5$, and consider the map

$$f(x) = \begin{cases} rx, & 0 \le x \le 1/2 \\ r - rx, & 1/2 \le x \le 1 \end{cases}.$$

(a) Why is it called a "tent map?"

(b) Find all the fixed points and classify the stability.

(c) Show that the map has a period-2 orbit. Is it stable or unstable?

(d) Can you find any period-3 points? How about period-4? If so, are the corresponding periodic orbits stable or unstable?

Problem 3

For $f : \mathbb{R} \to \mathbb{R}$ a C^1 map inducing a dynamical system on the real line, recall that the Lyapunov exponent at x is given by $\lambda_x(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f'(f^k(x))|$.

Compute the Lyapunov exponent for the linear map $f(x) = rx$ at any x.

Problem 4

Suppose that x^* is a stable fixed point ($f(x^*) = x^*$, $|f'(x^*)| < 1$). Then show that for any x whose orbit converges to x^*, $\lambda_x = \log|f'(x^*)|$. In particular, Lyapunov exponents are negative in the basins of attraction of stable fixed – and, more generally, periodic – points.

You may use the following analysis result as a black-box theorem: if a sequence of real numbers $\{y_n\}_{n\geq 0}$ converges to y, then $\lim_{n\to\infty} \frac{y_0+\cdots+y_{n-1}}{n} = y$.

Problem 5

Suppose that two dynamical systems $f, g : \mathbb{R} \to \mathbb{R}$ are conjugate via a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ (i.e. such that $h'(x)$ exists for all x, is continuous and nowhere zero), i.e. $f \circ h = h \circ g$. Also assume that there is a constant $C > 0$ such that $|h'(x)| \leq C$ for all x. Then show that for any $x \in \mathbb{R}$, $\lambda_x(g) = \lambda_{h(x)}(f)$.

This shows that Lyapunov exponents are "invariant" under C^1-conjugacy. In particular, if two systems have different Lyapunov exponents, they cannnot be C^1-conjugate.

Attendance