Teaching Assitant

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Office hours:

Mondays 11am-12pm - ARC 116
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Analyze the following equation graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

$$\dot{x} = (x^2 - 1)(x^2 - 4)$$

Problem 2

For the following system $\dot{x} = u(x, y)$, $\dot{y} = v(x, y)$, find the fixed points. Then sketch the nullclines, the vector field and a plausible phase portrait.

$$\dot{x} = y, \; \dot{y} = x(1 + y) - 1$$

Explore the non-hyperbolic fixed point $(0,0)$ of $\dot{x} = -x$, $\dot{y} = ay^2$.

(a) Show that for every $a \in \mathbb{R}$, $(x, y) = (0,0)$ is a fixed point, and the linearized system has the phase portrait of a "line of fixed points."

(b) Show that the nonlinear dynamics depends on a value of a, and may be either a "line of fixed points," a "sink," or a "saddle."

Problem 4

Consider the system $\dot{r} = r^4 - 8$, $\dot{\theta} = 1$.

(b) Determine the circle of maximum radius r_1, centered on the origin, such that $\dot{r} > 0$ whenever $r = r_1$.

(c) Determine the circle of minimum radius r_2, centered on the origin, such that $\dot{r} < 0$ whenever $r = r_2$.

(d) Use Poincaré-Bendixson to deduce that the system has a limit cycle in the trapping region $r_1 \leq r \leq r_2$.

(e) Why is $(0,0)$ the only possible fixed point?

Theorem 1 (Poincaré-Bendixson theorem, p205)

In the setting above, suppose that R is a closed bounded subset of the plane, and a trajectory $C = \{\mathbf{X}(t, x_0) \,:\, t \in \mathbb{R}\}$ starts in R and stays in R at all times. If R does not contain any fixed point, then either C is a closed orbit, or it spirals towards a closed orbit as $t \to \infty$. In either case R contains a closed orbit.

Problem 5

This problem aims at computing the Poincaré return map of the system $\dot{r} = ar^2$ for $a > 0$, mapping $x > 0$ to $P(x)$, the first crossing of the trajectory of $(x, 0)$ with the positive real axis.

(a) Given $r_0 > 0$, give the solution to the ODE $\dot{r} = ar^2$ with initial condition $r(0) = r_0$, denote it $r(t; r_0)$.

(b) Briefly justify why $P(x) = r(2\pi; x)$ for any $x > 0$, and write its expression.

Attendance