Teaching Assitant

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Office hours:

Mondays 11am-12pm - ARC 116
Thursdays 2:30-3:30pm - ARC 116

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Problem 1

(a) Show that if $S \subseteq \mathbb{R}$ is finite, then it has no limit points.

(b) Address the distance between the limit set of a sequence $\{u_n\}_{n \geq 0}$ and the limit set of its associated set $\{u_n : n \geq 0\}$. Consider the examples $u_n = \frac{1}{n}$ and $u_n = (-1)^n$.

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Definition: Limit Point of a (Real) Set

We say that y is a limit point of the set $S \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists x \in S \text{ such that } x \neq y \text{ and } \operatorname{dist}(x, y) < \varepsilon.$$

- What does S being finite tell use about the distance between points $x \in S$?
- How do we choose epsilon to show that this is true?

(b) Address the distance between the limit set of a sequence $\{u_n\}_{n\geq 0}$ and the limit set of its associated set $\{u_n : n \geq 0\}$. Consider the examples $u_n = \frac{1}{n}$ and $u_n = (-1)^n$.

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Definition: Limit Point of a (Real) Sequence

We say that y is a limit point of real-valued sequence $\{x_n\}_{n\geq 0}$ if

$$\forall \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists n \geq N \text{ such that } \operatorname{dist}(x_n, y) < \varepsilon.$$

- What are the limit points of these two sequences?
- Are those elements present in the respective sets $\{u_n : n \geq 0\}$?

Problem 2

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Definition: Density (in $\mathbb{R}$)

For $A, B \subseteq \mathbb{R}$, we say that A is dense in B if every point in B is a limit point of A.

- In plain language, what does it mean to be dense?
- Letting $A = \mathbb{Q}$ and $B = \mathbb{R}$, what is $B \setminus A$ (i.e. $\mathbb{R} \setminus \mathbb{Q}$)?

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- Take $y \in \mathbb{R} \setminus \mathbb{Q}$, and consider $(y - \varepsilon, y + \varepsilon)$ for any $\varepsilon > 0$. How do we know there exists $q \in \mathbb{Q}$ such that $q \in (y - \varepsilon, y + \varepsilon)$?

Problem 3

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The Doubling Map

Let $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ be given by $f(\theta) = 2\theta \pmod{2\pi}$.

- The point θ is k-periodic if $f^k(\theta) = \theta$.

Exercise

Solve $f^k(\theta) = \theta + 2p\pi$ for θ, where $p \in \mathbb{Z}$.

Exercise

Compute $\mathrm{Per}_2(f)$ and $\mathrm{Per}_3(f)$.

- Describe the behavior of k-periodic points on the unit circle.

Problem 4

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Definition: Metric Space

A metric space is a set X equipped with a distance function $d : X \times X \to [0,\infty)$ satisfying

1. $d(x,x) = 0$ for every $x \in X$,
2. $d(x,y) = d(y,x)$ for every $x,y \in X$,
3. and $d(x,y) \leq d(x,z) + d(z,y)$ for every $x,y,z \in X$.

Definition: Open Sets (in Metric Spaces)

We say that a set $U \subseteq X$ is open if for each point$x \in X$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

- What does the union of open sets do to their combined "size" (i.e. distance from interior points to the boundary)?
- Given any $x \in \bigcup_\alpha U_\alpha$, how do we go about picking ε?

Problem 5

Recall the definition of $F_\mu(x) = \mu x(1-x)$ (logistic map) for $\mu > 0$ and $Q_c(x) = x^2 + c$ (quadratic map) for $c \in \mathbb{R}$. Both families undergo "similar" bifurcation patterns as μ or c vary. The main reason is that they are conjugate via a linear transformation.

(a) Show that the map $h(x) = -4x + 2$ conjugates Q_{-2} and F_4 in the sense that $Q_{-2} \circ h = h \circ F_4$.

(b) More generally, for any $\mu > 0$, there exists a map $h(x) = ax + b$ which conjugates F_μ with Q_c, for some real numbers a, b, c. Find a, b, c by enforcing the equality $Q_c \circ h = h \circ F_\mu$.

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- Compute $Q_c(h(x))$ and $h(F_\mu(x))$.
- Observe that we have control over choosing a and b. If $Q_c(h(x)) = h(F_\mu(x))$, what choice for b allows simplification? Once simplified, can we repeat the process for a?
- Once we know a and b, what is the relation between μ and c that makes the equality true?

Attendance